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Masayuki Oikawa

Abstract

Some contributions to the theory of nonlinear waves are presented. Chapters 2 and 3 are concerned with the reductive perturbation method and chapters 4 and 5 the inverse scattering method. In chapter 2, the reductive perturbation method for weakly dispersive or dissipative systems is extended to multi-wave systems. First, the new perturbation method is developed for the head-on collision of two solitary waves in the Boussinesq equation. In the first approximation, the solution is described by a superposition of two solitary waves. The second order approximation gives a small correction where the two waves overlap one another. The method is extended to the system in which there exist n "quasi-simple" waves (the simple waves under the influence of dispersion or dissipation). The possibility that n quasi-simple waves which are governed by their respective Korteweg-de Vries or Burgers equation can be superposed to describe nonlinear systems is studied. Applications to ion acoustic waves in collisionless plasmas and shallow water waves are discussed. In particular, the phase shifts in the head-on collision of two solitary waves are calculated.

In chapter 3, the reductive perturbation method for strongly dispersive systems is extended to the system in which two modulated plane waves interact each other. The interaction of two envelope solitons is examined. It is shown that they pass through each other without change of amplitude and velocity.

In chapter 4, by following Ablowitz, Kaup, Newell and Segur's method, what class of nonlinear evolution equations can be solved by means of the third order eigenvalue problem. Some interesting examples are *is considered*

presented. One of them is a model equation for the interaction between Langmuir waves and ion sound waves in a plasma and has a solitary wave solution — a sonic-Langmuir soliton.

In chapter 5, this coupled system is solved by the inverse scattering method. The formation and the interaction of sonic-Langmuir solitons are studied. It is shown that the solitons preserve their identities through the interaction between them and that there exists an infinite number of constants of motion.

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Chapter 1

Introduction

In the last fifteen years, nonlinear wave propagation phenomena have been intensively studied and our understanding of them has been rapidly deepened. New asymptotic methods such as the Whitham method and the reductive perturbation method were developed for nonlinear wave equations. From detailed studies of properties of the Korteweg-de Vries equation and its solutions, the concept of solitons was introduced and the inverse scattering method for solving exactly the initial-value problem was discovered. These have been found to be available to many other nonlinear wave equations.

The present thesis is mainly concerned with the reductive perturbation method and the inverse scattering method. Therefore, in this chapter, they are outlined as preliminaries for the subsequent chapters together with the related subjects.

§ 1.1 Reductive Perturbation Method

In general, nonlinear wave propagation is described by a rather complicated system of equations. Therefore, in the initial stage of investigation, it is desired that the system of equations is simplified without loss of essential features. The reductive perturbation method is an asymptotic one which enables us to reduce a general nonlinear system to a single tractable nonlinear equation describing an asymptotic field of the system. This method for the propagation of long waves was established by Taniuti and Wei [1] in a general form applicable to both dispersive and dissipative systems.

The system of equations which they considered is

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + \left\{ \sum_{\beta=1}^s \prod_{\alpha=1}^p (H_{\alpha}^{\beta} \frac{\partial}{\partial t} + K_{\alpha}^{\beta} \frac{\partial}{\partial x}) \right\} U = 0, \quad (p \geq 2) \quad (1.1)$$

where U is a column vector with n components u_1, u_2, \dots, u_n , ($n \geq 2$) and A , H_{α}^{β} and K_{α}^{β} are $n \times n$ matrices which are functions of U . The expansion about a constant solution $U^{(0)}$ in a small parameter ε , that is,

$$U = U^{(0)} + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)} + \dots, \quad (1.2)$$

is assumed. It is also assumed that $A_0 (\equiv A(U^{(0)}))$ has at least one real and non-degenerate eigenvalue λ_0 . Let us introduce the stretched coordinates,

$$\xi = \varepsilon^a (x - \lambda_0 t), \quad (1.3a)$$

$$\tau = \varepsilon^{a+1} t, \quad (1.3b)$$

$$a = (p-1)^{-1},$$

then the system (1.1) can be reduced to

$$\frac{\partial f}{\partial \tau} + \alpha f \frac{\partial f}{\partial \xi} + \beta \frac{\partial^p f}{\partial \xi^p} = 0, \quad (1.4)$$

under the boundary condition for $x \rightarrow \infty$, $U^{(1)} \rightarrow 0$ [1]. Here $U^{(1)}$ is given by

$$U^{(1)} = fR,$$

with R the right eigenvector of A_0 , i.e.,

$$A_0 R = \lambda_0 R,$$

while the coefficients α and β are given by the equations

$$\alpha = (L, (R\nabla)A_0 R) / (L, R), \quad (1.5a)$$

$$\beta = (L, \sum_{\beta=1}^S \prod_{\alpha=1}^p (K_{\alpha 0}^{\beta} - \lambda_0 H_{\alpha 0}^{\beta}) R) / (L, R), \quad (1.5b)$$

where L stands for the left eigenvector, $(\ , \)$ the inner product and

$$(R\nabla)A_0 \equiv \sum_k r_k \left(\frac{\partial A}{\partial u_k} \right)_{U=U^{(0)}}.$$

If α or β vanishes, the stretched coordinates (1.3) must be modified appropriately and so the equation (1.4) [1]-[3].

The equation (1.4) takes the form of the Burgers equation for $p=2$ and that of the Korteweg-de Vries (KdV) equation for $p=3$. These equations were also obtained by Su and Gardner [4] from systems of conservation laws.

The stretched coordinates (1.3) for $p=3$ were first introduced by Gardner and Morikawa [5], who showed the KdV equation to describe the long time asymptotic behaviour of collision-free hydromagnetic waves with a small but finite amplitude and a long wavelength. Their requirement to obtain the KdV equation is that it includes both a linearized time-dependent asymptotic motion and a pulse-like weak nonlinear steady state which is formed owing to the balance between nonlinearity and dispersion. According to this requirement, (1.3) may be derived in the following way. Linearization of (1.1) about $U^{(0)}$ and successive approximation give rise to dispersion relation for long waves

$$\frac{\omega}{k} = \lambda_0 + i^{p-1} \beta k^{p-1} + \dots, \quad k \ll 1, \quad (1.6)$$

where k and ω are wave number and wave frequency respectively and β is given

by (1.5b). On the other hand, the characteristics deviate from the parallel straight lines in the linear approximation owing to the nonlinear effect, that is,

$$\frac{dx}{dt} = \lambda_0 + \varepsilon \lambda_1 + \dots, \quad (1.7)$$

where λ_1 is proportional to α given by (1.5a) and the last term in (1.1) has been left out of consideration. Comparison of (1.6) and (1.7) shows that in order for the dispersive effect and the nonlinear effect to be comparable in magnitude, we must take as

$$k \sim \varepsilon^a, \quad a = (p-1)^{-1}. \quad (1.8)$$

This means that the characteristic length of the spatial variation of the wave in the wave frame is of order of ε^{-a} . From this, (1.3a) is derived. On the other hand, by using (1.6), we obtain

$$kx - \omega t = k(x - \lambda_0 t) - i^{p-1} \beta k^p t. \quad (1.9)$$

This means that the characteristic time scale of the wave in the wave frame is of order of k^{-p} ($= \varepsilon^{-(a+1)}$). Therefore, (1.3b) is derived. As seen from the above discussion, the equation (1.4) describes the long-time asymptotic behaviour of the mode corresponding to the eigenvalue λ_0 under the influence of the weak nonlinearity and the weak dispersion (or dissipation). From the standpoint of the theory of nonlinear hyperbolic waves, (1.4) can be also considered to describe the behaviour of a simple wave under the influence of the dispersion (or dissipation), which may be called a "quasi-simple" wave. It is important to note the following : In order for a nonlinear wave system to be reduced to (1.4), it is only required that the

dispersive (or dissipative) effect and the nonlinear effect manifest themselves in the forms of (1.6) and (1.7) respectively.

The Burgers equation [6]

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \beta \frac{\partial^2 u}{\partial x^2}, \quad (\beta > 0) \quad (1.10)$$

is the simplest model equation to account for the formation of shock waves. The equation is derived from the system of hydrodynamic equations for a compressible, viscous and heat-conducting fluid [1],[7], and it is also used as a model of the one-dimensional turbulence [8]. A remarkable property of this equation is that it is reduced to the linear heat equation by the Hopf-Cole transformation [9],[10]. Because of this property, various questions can be investigated in great detail [11].

The KdV equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad (1.11)$$

was first derived by Korteweg and de Vries(1895) [12] for shallow water waves. This equation is very important, because in many dispersive systems, the dispersion relations for long waves take the forms of (1.6) with $p=3$. In fact, since 1960, many applications of the equation have been found [13]-[24].

The equation (1.11) has solitary wave solutions

$$u = 3a \operatorname{sech}^2 \left\{ \frac{\sqrt{a}}{\sqrt{4}} (x - x_0 - at) \right\}, \quad (a > 0) \quad (1.12)$$

which were found by Korteweg and de Vries [12] together with nonlinear periodic wave solutions named cnoidal waves by them. These solitary waves are those discovered experimentally by Scott Russell (1844) [25] and analyzed approximately by Boussinesq [26] and Rayleigh [27]. The solitary wave solutions (1.12) are fundamental quantities of the general solution to (1.11) and were named "solitons" by Zabusky and Kruskal [28] because of their resemblance to particles. The solitons with different amplitudes preserve their identities through collisional interaction between them. After the interaction, they are only shifted in their positions relative to where they would have been if no interaction had taken place. This behaviour of solitons was first found numerically by Zabusky and Kruskal [14],[28] and proved by Lax [29] for the case of two solitons. The corresponding result for N distinct solitons can be derived [30]-[33] by making use of the inverse scattering method — the method of exact solution discovered by Gardner, Greene, Kruskal and Miura [34] and described in the next section. Emergence of solitons from long wavelength sinusoidal initial data [14],[28] and from gaussian initial data [35] was also found through numerical computations. In general, the asymptotic state for $t \rightarrow \infty$ of a localized initial disturbance on $(-\infty, \infty)$ consists of a certain number (which may be zero) of solitons propagating to the right and a dispersing oscillatory state propagating to the left.

The results of the experiments on ion-acoustic waves [36] and water waves [37],[38] agree well with the predictions of the KdV equation in essential features.

The equation (1.4) has been derived for long waves. Therefore, it is not applicable to the problem in which the characteristic wave number is large, for example, to the problem of slow modulation of a plane wave.

It is well known that in nonlinear dispersive media, there do exist nonlinear plane waves of permanent type which are formed owing to the balance between nonlinear steepening and dispersion in common with the above-mentioned solitary waves. The Stokes wave [39] is a typical example of such nonlinear plane waves.

The reductive perturbation method for slow modulation of weak nonlinear plane waves was established by Taniuti and Washimi [40] for a special example and was generalized by Taniuti and Yajima [41] to a wide class of nonlinear wave systems. The basic equation which they considered is

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + B = 0, \quad (1.13)$$

where U and A are taken as in (1.1), and B is a column vector with n components and is a function of U . The following expansion of U about a constant solution $U^{(0)}$ satisfying $B(U^{(0)}) = 0$ in terms of a small parameter ε is assumed;

$$U = U^{(0)} + \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} U^{(\alpha)}, \quad (1.14a)$$

$$U^{(\alpha)} = \sum_{\ell=-\infty}^{\infty} U_{\ell}^{(\alpha)}(\xi, \tau) \exp i \ell (kx - \omega t), \quad (1.14b)$$

$$U_{\ell}^{(\alpha)} = U_{\ell}^{(\alpha)*}, \quad (1.14c)$$

where ξ and τ are stretched coordinates given by

$$\xi = \varepsilon(x - \lambda t), \quad (1.15a)$$

$$\tau = \varepsilon^2 t, \quad (1.15b)$$

and (1.14c) is the reality condition for $U^{(\alpha)}$ and the asterisk denotes the complex conjugate, and k and ω are respectively the central wave number and frequency of the modulated wave under consideration. Substitution of the expansion (1.14) into (1.13) leads to the following results : First, for k and ω satisfying the dispersion relation

$$\det W_l = 0, \quad W_l \equiv -i\omega I + ikA_0 + \nabla B_0,$$

$U^{(1)}_{\ell=0}$ ($\ell = \pm 1$), where $A_0 \equiv A(U^{(0)})$ and $(\nabla B_0)_{ij} = (\partial B_i / \partial u_j)_{U=U^{(0)}}$.

Secondly, λ must be equal to the group velocity $\partial\omega/\partial k$. Thirdly,

$$i \frac{\partial \varphi}{\partial \tau} + p \frac{\partial^2 \varphi}{\partial \xi^2} + g|\varphi|^2 \varphi = 0, \quad (1.16)$$

$$U_i^{(u)} = \varphi R,$$

where R is the right eigenvector, i.e., $W_l R = 0$, and $p = \frac{1}{2} \frac{\partial^2 \omega}{\partial k^2}$.

As the explicit form of q is a little lengthy, it has not been written down.

The equation (1.16) may be obtained by the following heuristic argument. Let us consider first linear wave modulation. A slowly modulated linear wave is written as a Fourier integral,

$$u = \int u(k) \exp[i(kx - \omega t)] dk, \quad (1.17)$$

where the spectrum $u(k)$ has a peak at k_0 with narrow width $\delta k \sim \varepsilon$.

Then, the dispersion relation may be expanded about k_0 , i.e.,

$$\omega(k) = \omega_0 + \omega'_0 \delta k + \frac{1}{2} \omega''_0 \delta k^2 + \dots, \quad (1.18)$$

where the subscript denotes the value at k_0 and the prime denotes the differentiation with respect to k . Introduction of (1.18) into (1.17) yields

$$u \approx \exp[i(k_0 x - \omega_0 t)] \cdot \int u(k_0 + \delta k) \exp[i\{\delta k(x - \omega'_0 t) - \frac{1}{2}\omega''_0 \delta k^2 t\}] d\delta k. \quad (1.19)$$

The integral with respect to δk is a slowly varying function of $x - \omega'_0 t$ and t , and hence this equation expresses the slow modulation of the carrier plane wave. Moreover, if we introduce ξ and τ by (1.15) with ω'_0 as λ and write the envelope as φ , i.e.,

$$u \approx \varphi(\xi, \tau) \exp[i(k_0 x - \omega_0 t)], \quad (1.20)$$

it is readily seen that φ is a solution of the equation

$$i \frac{\partial \varphi}{\partial \tau} + \frac{\omega''_0}{2} \frac{\partial^2 \varphi}{\partial \xi^2} = 0. \quad (1.21)$$

In a nonlinear plane wave, the nonlinear effect manifests itself as the frequency shift. Since φ is a slowly varying function, the weak nonlinear effect for dispersion relation may be written in local as

$$\omega(k, |\varphi|^2) = \omega_0 + \omega'_0 \delta k + \frac{1}{2} \omega''_0 \delta k^2 - g |\varphi|^2 + \dots, \quad (1.22)$$

where in order for the third term and the fourth term in the right hand side of (1.22) to be comparable in magnitude, we are to take $|\varphi| \sim \varepsilon$. If we take this nonlinear correction into account, we at once obtain (1.16).

The equation (1.16) was also derived by Benney and Newell [42] and by Karpman and Krushkal [43] in different ways. It can describe a wide

class of physical phenomena — the modulational instability of water waves [44], the propagation of heat pulses in anharmonic crystals [45],[46], the herical motion of a very thin vortex filament [47], the nonlinear modulation of collisionless plasma waves [48] and the two-dimensional self-focusing of stationary light beams [49]-[51].

By the transformation $\varphi \rightarrow \rho u$, $\xi \rightarrow x$, $\rho \tau \rightarrow t$, (1.16) is reduced to

$$i \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \kappa |u|^2 u = 0, \quad (1.23)$$

where $\kappa = \rho g$. This is the Schrödinger equation with the nonlinear potential $-\kappa |u|^2$. Therefore, the equation of this type is called the nonlinear Schrödinger equation. The potential is attractive or repulsive, according as the sign of κ is positive or negative. As expected, the character of the solutions of (1.23) is governed by the sign of κ . The plane wave solution of (1.23) can be easily found;

$$u = u_0 \exp[i(\mu x - Et)], \quad (1.24)$$

where u_0 is a constant and

$$E = \mu^2 - \kappa u_0^2 \quad (1.25)$$

If $\mu = 0$, this solution corresponds to a nonlinear plane wave solution of the original system. The plane wave (1.24) is modulationally unstable if $\kappa > 0$, and conversely, it is modulationally stable if $\kappa < 0$ [41],[42]. That is, if $\kappa > 0$, the plane wave is unstable against the modulational perturbations with sufficiently large wavelength. The modulational instabili—

of nonlinear plane waves satisfying $\omega^2 g > 0$, in particular, that of the Stokes wave was predicted by Lighthill [52] and Whitham [53] by making use of Whitham's averaging technique [54] in which the local oscillations are averaged out so as to obtain the differential equations for slow variables such as amplitude and wave number. The modulational instability of the Stokes wave was also predicted by Benjamin and Feir [55],[56] in a different way and confirmed experimentally by them [55].

In the case $\kappa > 0$, owing to the balance between the diffusion of the wave packet due to dispersion and the self-trapping effect due to nonlinearity, the envelope solitary wave solutions of (1.23) are produced;

$$\sqrt{\frac{\kappa}{2}} u = a \operatorname{sech}[a(x-x_0-2vt)] \exp[ivx - i(v^2 - a^2)t + i\theta], \quad (1.26)$$

where a , v , x_0 and θ are real parameters. These solutions are called the "envelope solitons" because they are analogous in many respects to the solitons in the KdV equation. The numerical computations carried out by Karpman and Krushkal [43], Yajima and Outi [57] and Chu and Mei [58] suggest strongly the development of a modulated wave into the envelope solitons. The envelope solitons with different velocities preserve their amplitudes and velocities even after the interaction of them. This was found numerically by Yajima and Outi [57] for two envelope solitons and verified by Zakharov and Shabat [59] for N envelope solitons by means of the inverse scattering method. The equation (1.23) with positive κ is more complicated than the KdV equation in respect that there exist the solutions representing the interaction of a finite number of envelope solitons with identical velocities [59].

In the case $\kappa < 0$, the solitary wave solutions are also possible, and they are given by

$$\sqrt{\frac{-\kappa}{2}} u = [\mu - i\nu \tanh \nu(x-x_0-2\mu t)] \exp(ixu_0^2 t + i\theta), \quad \nu = \sqrt{\frac{(-\kappa)}{2} u_0^2 - \mu^2}, \quad (1.27)$$

and in this case,

$$|u|^2 = u_0^2 + \frac{2}{\kappa} \nu^2 \operatorname{sech}^2 \nu(x-x_0-2\mu t), \quad (1.28)$$

where μ , x_0 and θ are real parameter and it has been assumed that $|u|^2 \rightarrow u_0^2$ and $u_x \rightarrow 0$ as $x \rightarrow \pm\infty$. When $\mu = 0$, (1.27) represents a phase jump. The concept of soliton also applies to the solitary wave solutions (1.27) [60].

The reductive perturbation method has been extended so as to be applicable to weakly inhomogeneous systems [61],[62] and also to the Vlasov plasma [63]-[72]. The method was used to investigate the effect of a random field [73] and that of quasi-particles [74] on long waves with finite amplitude. The reductive perturbation method for wave modulation has been extended to weakly unstable (or dissipative) systems [62]. In fact, some generalized versions of (1.16) were obtained in hydrodynamic stability problems to describe the nonlinear evolution of unstable modes at a slightly super-critical value of the relevant parameter [75]-[77].

In chapters 2 and 3, a generalization of the reductive perturbation method to multi-wave systems is presented. The reductive perturbation method for long waves is concerned with the propagation of the single quasi-simple wave along one family of characteristic. Therefore, it is inapplicable to the problems such as the collision (head-on collision) of two ion-acoustic

solitary waves propagating in opposite directions each other. In chapter 2, with some modification of the stretched variables (1.3), the method is extended to be applicable to such problems, and then to the case that there exist n quasi-simple waves each of which propagates along one of n families of characteristics. It is shown that in some sense, a solution of (1.1) can be described as a superposition of n quasi-simple waves each of which is governed by the type of equation (1.4). This extended version of the method is applied to the head-on collision of two solitary waves in shallow water waves and in ion-acoustic waves. The phase shift of each solitary wave in the head-on collision is obtained.

The reductive perturbation method for nonlinear wave modulation in dispersive media is concerned with the self-modulation of a wave packet with small bandwidth about a central wave number and frequency. In chapter 3, this method is generalized so as to be able to deal with the mutual-interaction of two wave packets with group velocities different from each other by the order unity. It is shown that the two wave packets can be described by the respective nonlinear Schrödinger equations and except for phase changes in the envelopes and for those in the carrier waves, are unaffected by the mutual-interaction. In particular, the mutual-interaction of two envelope solitary waves with propagation velocities different from each other by the order unity is investigated.

§ 1.2 Inverse Scattering Method

One of the most remarkable results of recent study of nonlinear waves is the discovery of the inverse scattering method which is a method of exact solution for nonlinear evolution equations and the distinguished character of which is that it reduces the initial value problem of a nonlinear evolution equation to the direct and inverse problem of scattering of a certain linear operator. The method was first developed by Gardner, Greene, Kruskal and Miura (GGKM) [34] in order to solve the initial value problem of the KdV equation.

We consider the initial value problem of the KdV equation (1.11) on $-\infty < x < \infty$, with the initial data $u(x, t=0) = u_0(x)$ vanishing rapidly as $|x| \rightarrow \infty$. Motivated by Miura's transformation [78] relating solutions of (1.11) and (2.1), the modified KdV equation

$$\frac{\partial v}{\partial t} + v^2 \frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0, \quad (2.1)$$

which arises in lattice dynamics [13], [14] and plasma physics [3], [79], GGKM were led to consider the Schrödinger equation

$$-\frac{\partial^2 f}{\partial x^2} + g(x, t)f = \zeta^2 f, \quad g(x, t) = -\frac{1}{6}u(x, t) \quad (2.2)$$

in which $u(x, t)$ evolves according to the KdV equation (1.11).

First, we consider the direct and inverse scattering problem of the Schrödinger equation without time dependence,

$$-\frac{\partial^2 f}{\partial x^2} + g(x)f = \zeta^2 f, \quad (2.3)$$

where the condition [80]

$$\int_{-\infty}^{\infty} (1+|x|) |q(x)| dx < \infty \quad (2.4)$$

is assumed. The equation (2.3), subject to this condition, has solutions $\varphi(x, \zeta)$ and $\psi(x, \zeta)$, uniquely defined for all real ζ by the conditions

$$\varphi(x, \zeta) \rightarrow e^{-i\zeta x}, \quad x \rightarrow -\infty; \quad \psi(x, \zeta) \rightarrow e^{i\zeta x}, \quad x \rightarrow +\infty. \quad (2.5)$$

Here $\varphi(x, -\zeta) = [\varphi(x, \zeta)]^*$, $\psi(x, -\zeta) = [\psi(x, \zeta)]^*$ are satisfied because $q(x)$ is real. Since for $\zeta \neq 0$, $\psi(x, \zeta)$ and $\psi(x, -\zeta)$ are linearly independent solutions of (2.3), $\varphi(x, \zeta)$ can be written as

$$\varphi(x, \zeta) = a(\zeta) \psi(x, -\zeta) + b(\zeta) \psi(x, \zeta), \quad (2.6)$$

where the coefficients $a(\zeta)$ and $b(\zeta)$ satisfy the relations

$$a(\zeta) = [a(-\zeta)]^*, \quad b(\zeta) = [b(-\zeta)]^*, \quad |a|^2 = 1 + |b|^2. \quad (2.7)$$

In addition,

$$a(\zeta) = \frac{1}{2i\zeta} W(\psi(x, \zeta), \varphi(x, \zeta)), \quad (2.8)$$

where $W(f, g) \equiv f_x g - f g_x$ is the Wronskian. $1/a(\zeta)$ and $b(\zeta)/a(\zeta)$ are called the transmission and reflection coefficient respectively. Under the condition (2.4), the solutions $\varphi(x, \zeta)$ and $\psi(x, \zeta)$ and the coefficient $a(\zeta)$ can be continued analytically into the upper half ζ -plane ($\text{Im}(\zeta) > 0$) and have the following asymptotic properties for large $|\zeta|$, $\text{Im}(\zeta) > 0$;

$$\varphi(x, \zeta) e^{i\zeta x} = 1 + O(\zeta^{-1}), \quad \psi(x, \zeta) e^{-i\zeta x} = 1 + O(\zeta^{-1}),$$

$$a(\zeta) = 1 + O(\zeta^{-1}). \quad (2.9)$$

In the upper half ζ -plane, $a(\zeta)$ can have only a finite number of zeros.

We assume that $a(\zeta)$ has there N zeros and designate them $i\eta_n$,

$\text{Im}(i\eta_n) > 0$ ($n=1, 2, \dots, N$). For $\zeta = i\eta_n$, $\varphi(x, \zeta)$ and $\psi(x, \zeta)$ are linearly dependent, i.e.,

$$\varphi(x, i\eta_n) = \theta_n \psi(x, i\eta_n), \quad (2.10)$$

and thus $\psi(x, i\eta_n)$ decreases exponentially as $|x| \rightarrow \infty$, that is,

$\psi(x, i\eta_n)$ is the eigenfunction of a bound state. Therefore,

$i\eta_n$ ($n=1, 2, \dots, N$) is the discrete spectrum and η_n is real and positive and then $\psi(x, i\eta_n)$ is also real. $i\eta_n$ is a simple zero of $a(\zeta)$

because the following relation can be verified :

$$i \dot{a}(i\eta_n) = \int_{-\infty}^{\infty} \varphi(x, i\eta_n) \psi(x, i\eta_n) dx = \theta_n \int_{-\infty}^{\infty} \psi(x, i\eta_n)^2 dx \neq 0 \quad (2.11)$$

where the dot over a denotes the derivative with respect to ζ .

The normalization constant $c_n (> 0)$ is defined by the relation

$$c_n^{-2} = \int_{-\infty}^{\infty} \psi(x, i\eta_n)^2 dx, \quad (2.12)$$

and then from (2.11) we obtain

$$c_n^2 = -i \frac{\theta_n}{\dot{a}(i\eta_n)}. \quad (2.13)$$

If $q(x)$ vanishes sufficiently rapidly as $|x| \rightarrow \infty$ so that $b(\zeta)$ is also defined at $\zeta = i\eta_n$, in view of (2.6) and (2.10),

$$b_n = b(i\eta_n). \quad (2.14)$$

Therefore, in this case, c_n is related to the residue of $b(\zeta)/a(\zeta)$ at the pole $\zeta = i\eta_n$. The set $S \{i\eta_n, c_n (n=1, 2, \dots, N), b(\zeta)/a(\zeta) (\zeta: \text{real})\}$ is called the scattering data. In this way the scattering data is determined completely from the potential.

The inverse problem, that is, the problem of the reconstruction of the potential from the scattering data was investigated by Gel'fand and Levitan [81], Marchenko [82] and Kay and Moses [83] (see also Faddeev [80]). According to them, the inverse problem can be uniquely solved. Let us define, in terms of the scattering data,

$$B(z) = \sum_{n=1}^N c_n^2 e^{-\eta_n z} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b(\zeta)}{a(\zeta)} e^{i\zeta z} d\zeta. \quad (2.15)$$

If we solve the Gel'fand-Levitan (or Marchenko) equation

$$K(x, y) + B(x+y) + \int_x^{\infty} K(x, s) B(s+y) ds = 0, \quad y > x, \quad (2.16)$$

subject to the condition $K(x, y) \rightarrow 0$ as $y \rightarrow \infty$, the potential $q(x)$ is given by the relation

$$q(x) = -2 \frac{d}{dx} K(x, x). \quad (2.17)$$

Now we proceed to consider the equation (2.2). By substituting $u(x, t)$ of (2.2) into the KdV equation (1.11), we obtain, for the time development of f ,

$$\frac{\partial \zeta^2}{\partial t} f^2 + \frac{\partial}{\partial x} \left(f \frac{\partial R}{\partial x} - \frac{\partial f}{\partial x} R \right) = 0, \quad (2.18a)$$

$$R \equiv \frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} - 3(\zeta + \zeta^2) \frac{\partial f}{\partial x}. \quad (2.18b)$$

If f is the eigenfunction of a bound state, integration of (2.18a) over the interval $(-\infty, \infty)$ yields $\partial \zeta^2 / \partial t = 0$, that is, the discrete spectrum of (2.2), $i\omega_n$ ($n=1,2, \dots, N$) remains invariant when $u(x,t)$ evolves according to the KdV equation (1.11). This was the first remarkable discovery of GGKM. Then, by making use of (2.2), (2.18a) can be reduced to

$$R \equiv \frac{\partial f}{\partial t} + \frac{\partial^3 f}{\partial x^3} - 3(\zeta + \zeta^2) \frac{\partial f}{\partial x} = C f + D f \int_0^x \frac{dx}{f^2}, \quad (2.19)$$

where C and D are independent of x . For the continuous spectrum, ζ can be assumed to remain invariant; hence f again satisfies (2.19). In view of (2.5) and (2.6), $\varphi(x, \zeta, t)$ behaves as $x \rightarrow \pm\infty$ in the following way:

$$\varphi(x, \zeta, t) \cong e^{-i\zeta x}, \quad x \rightarrow -\infty, \quad (2.20a)$$

$$\varphi(x, \zeta, t) \cong a(\zeta, t) e^{-i\zeta x} + b(\zeta, t) e^{i\zeta x}, \quad x \rightarrow +\infty. \quad (2.20b)$$

Substitution of these asymptotic forms into (2.19) yields

$$C = 4i\zeta^3, \quad D = 0, \quad (2.21)$$

and

$$\frac{\partial a}{\partial t} = 0, \quad \frac{\partial b}{\partial t} = 8i\zeta^3 b. \quad (2.22)$$

If we utilize the relations (2.13) and (2.14), owing to (2.22), we can obtain

$$\frac{\partial c_n}{\partial t} = 4\gamma_n c_n. \quad (2.23)$$

This agrees with the result obtained in a different way [34] which is also applicable to the case that $b(i\gamma_n)$ is not defined. Thus, the time evolution of the scattering data is given by the equations

$$\gamma_n(t) = \gamma_n(0), \quad (n = 1, 2, \dots, N), \quad (2.24a)$$

$$c_n(t) = c_n(0) e^{4\gamma_n^3 t}, \quad (n = 1, 2, \dots, N), \quad (2.24b)$$

$$a(\zeta, t) = a(\zeta, 0), \quad (2.24c)$$

$$b(\zeta, t) = b(\zeta, 0) e^{8i\zeta^3 t}. \quad (2.24d)$$

It is important to observe that the time evolution of the scattering data is determined without a knowledge of $u(x, t)$. This observation was the second remarkable discovery of GGKM.

The procedure of the inverse scattering method is summarized as follows. For given initial data $u_0(x)$, we first solve the direct scattering problem of (2.3) with the potential $q(x) = -u_0(x)/6$ from which the scattering data at $t=0$ is determined. The scattering data at any later time is given by (2.24) and

$$B(z, t) = \sum_{n=1}^N c_n^2(0) e^{8\gamma_n^3 t - \gamma_n z} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b(\zeta, 0)}{a(\zeta, 0)} e^{8i\zeta^3 t + i\zeta z} d\zeta. \quad (2.25)$$

This defines the Gel'fand-Levitan equation (2.16) for all time t and its solution $K(x,y,t)$ leads to the desired solution of the KdV equation (1.11)

$$u(x,t) = -6g(x,t) = 12 \frac{d}{dx} K(x,x,t).$$

Here it should be noted that both equations (2.3) and (2.16) are linear and t appears only parametrically throughout.

The special case, in which $b(\zeta, 0) = 0$ for all real ζ , was investigated by several authors [30]–[33]. In this case, the Gel'fand-Levitan equation can be reduced to the system of linear algebraic equations, so that it can be easily solved to yield the so-called N -soliton solution. For $N=1$, the solution is given by

$$u(x,t) = 12\eta_1^2 \operatorname{sech}^2\{\eta_1(x - 4\eta_1^2 t - \delta_1)\}, \quad \delta_1 = \frac{1}{2\eta_1} \log \frac{C_1(0)}{2\eta_1}. \quad (2.26)$$

This is the soliton solution of (1.11) with the amplitude $12\eta_1^2$ and the velocity $4\eta_1^2$, that is, a zero of $a(\zeta)$ corresponds to a soliton. For arbitrary N , the asymptotic behaviours of the solution for $t \rightarrow \pm\infty$ are given by

$$u(x,t) \rightarrow \sum_{n=1}^N 12\eta_n^2 \operatorname{sech}^2\{\eta_n(x - 4\eta_n^2 t - \delta_{n\pm})\}, \quad t \rightarrow \pm\infty \quad (2.27a)$$

$$\rightarrow 12\eta_n^2 \operatorname{sech}^2\{\eta_n(x - 4\eta_n^2 t - \delta_{n\pm})\}, \quad t \rightarrow \pm\infty \quad (x - 4\eta_n^2 t : \text{fixed}), \quad (2.27b)$$

where

$$\delta_{n+} = \frac{1}{2\eta_n} \log \frac{C_n(0)}{2\eta_n} + \frac{1}{\eta_n} \sum_{m=1}^{n-1} \log \frac{\eta_m - \eta_n}{\eta_m + \eta_n}, \quad \delta_{n-} = \frac{1}{2\eta_n} \log \frac{C_n(0)}{2\eta_n} + \frac{1}{\eta_n} \sum_{m=n+1}^N \log \frac{\eta_n - \eta_m}{\eta_n + \eta_m}, \quad (2.28)$$

and $\eta_1 > \eta_2 > \dots > \eta_N$ has been assumed. The solution breaks up, as $t \rightarrow \pm\infty$, into the same solitons each of which corresponds to one of the zeros of $a(\zeta)$. The n -th soliton is only shifted, owing to the interactions with the other solitons, by $\Delta\delta_n = \delta_{n+} - \delta_{n-}$, as it travels from $t \rightarrow -\infty$ to ∞ .

On the other hand, for $b(\zeta, 0) \neq 0$, the continuous spectrum also contributes to the solution and corresponds to the nonsoliton part of it. The nonsoliton part has an oscillatory structure, and disperses to the left and decays to zero in time asymptotically [84],[85]. In this general case, as $t \rightarrow \infty$, in the region $x > \varepsilon t$ (ε is an arbitrary positive number), the solution is again approximated by the sum of the solitons each of which corresponds to one of the zeros of $a(\zeta)$ [86]. These results support the numerical study by Berezin and Karpman [35] and the analytical and numerical study by Zabusky [87].

The KdV equation has an infinite number of conservation laws of the type

$$\frac{\partial T_n}{\partial t} + \frac{\partial X_n}{\partial x} = 0, \quad (n=1, 2, 3, \dots) \quad (2.29)$$

from which we can obtain the constants of motion $I_n = \int_{-\infty}^{\infty} T_n dx$ if X_n vanishes as $|x| \rightarrow \infty$, and the first three examples of I_n are given by

$$I_1 = \int_{-\infty}^{\infty} u dx, \quad I_2 = \int_{-\infty}^{\infty} \frac{u^2}{2} dx, \quad I_3 = \int_{-\infty}^{\infty} \left(\frac{u^3}{3} - u_x^2 \right) dx. \quad (2.30)$$

This fact was first shown by Miura, Gardner and Kruskal [88] through Miura's transformation and Galilean transformation. The constants of motion I_n ($n=1, 2, \dots$) can be also derived by the method based on the time invariance of $a(\zeta)$ [89].

One of the key points for success of the inverse scattering method is that the eigenvalues of (2.2) remain invariant as $u(x,t)$ evolves according to (1.11). The generalization of the interrelation between (1.11) and (2.2) was provided by Lax [29]. According to him, the method may be applicable to the (nonlinear) evolution equation

$$\frac{\partial u}{\partial t} = K[u], \quad (2.31)$$

which can be represented in the form

$$\frac{\partial L}{\partial t} = [B, L] = BL - LB, \quad (2.32)$$

where L and B are linear differential operators including the function (or vector) $u(x,t)$ in the coefficients and $\partial/\partial t$ refers to differentiating u with respect to t in the expression for L . Consider the eigenvalue problem

$$L\psi = \zeta\psi. \quad (2.33)$$

If ψ evolves according to

$$\frac{\partial \psi}{\partial t} = B\psi, \quad (2.34)$$

then by differentiating (2.33) with respect to t , it follows that $\partial\zeta/\partial t = 0$. In this case, the initial value problem of (2.31) can be reduced to the direct and inverse scattering problem of (2.33). The time evolution of the scattering data is found from (2.34). It should be noted that B is only determined up to an operator commuting with L . We will take advantage of this fact in determining the time evolution of the scattering data.

Following Lax's approach, Zakharov and Shabat [59] succeeded in

finding L and B appropriate to nonlinear Schrödinger equation (1.23) with positive κ which is the second example solved by means of the method.

These operators are given by

$$L = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x} - i \begin{pmatrix} 0 & g \\ g^* & 0 \end{pmatrix}, \quad (2.35a)$$

$$B = 2i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial^2}{\partial x^2} - 2i \begin{pmatrix} 0 & g \\ g^* & 0 \end{pmatrix} \frac{\partial}{\partial x} - i \begin{pmatrix} -|g|^2 & g_x \\ g_x^* & |g|^2 \end{pmatrix}, \quad g = \sqrt{\frac{\kappa}{2}} u, \quad (2.35b)$$

Assuming that $|u|$ vanishes rapidly as $x \rightarrow \pm\infty$, they solved the direct and inverse scattering problem of the operator (2.35a). The discrete eigenvalue ζ_n (which is a zero of $a(\zeta)$ analogous to $a(\zeta)$ in the scattering problem of the Schrödinger operator and in general, complex) corresponds to a (envelope) soliton and its velocity and amplitude are determined by $\text{Re}(\zeta_n)$ and $\text{Im}(\zeta_n)$, respectively. The behaviours of a N-soliton (with distinct velocities) solution is analogous to the case in the KdV equation except for presence of the phase shifts in the carrier waves. A finite number of the solitons with the same velocity do not separate but form a bound state of the solitons. Then $|u(x,t)|$ is periodic in time in the frame moving with the velocity. They also obtained an infinite number of constants of motion for (1.23) with $\kappa > 0$. The behaviours of solutions for various initial conditions were investigated in detail by Satsuma and Yajima [90] on the basis of the method of Zakharov and Shabat.

Wadati [91] and Tanaka [92] noted that the modified KdV equation (2.1) can be also related to the operator (2.35a). The N-soliton solution was obtained by them. The bound state of solitons was also found.

An infinite number of conservation laws for this equation had been previously found in [88].

The same type of operator was again used by Ablowitz, Kaup, Newell and Segur (AKNS) [93] to apply the inverse scattering method to the sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0 \quad (2.36a)$$

or

$$\frac{\partial^2 u}{\partial x \partial t} = \sin u, \quad (2.36b)$$

which arises in many branches of physics [94]-[102] and also in differential geometry [103]. The solitary wave solution of the sine-Gordon equation corresponds to a rotation in u by 2π as x goes from $-\infty$ to ∞ . The pulse corresponding to a positive sense of rotation is sometimes called "kink" and that corresponding to a negative sense of rotation "antikink". Particular solutions including those representing "kink-kink" and "kink-antikink" collisions had been already found [102],[104]-[107]. The sine-Gordon equation (2.36b) was also investigated in detail by Takhtadzhyan [108] by means of the inverse scattering method.

AKNS [109] found that many nonlinear evolution equations can be solved by the generalized Zakharov-Shabat eigenvalue problem:

$$\frac{\partial v_1}{\partial x} + i\zeta v_1 = g(x, t) v_2, \quad (2.37a)$$

$$\frac{\partial v_2}{\partial x} - i\zeta v_2 = r(x, t) v_1. \quad (2.37b)$$

Choose the time dependence of the eigenfunctions v_1 and v_2 to be

$$\frac{\partial v_1}{\partial t} = A(x, t, \zeta) v_1 + B(x, t, \zeta) v_2, \quad (2.38a)$$

$$\frac{\partial v_2}{\partial t} = C(x, t, \zeta) v_1 - A(x, t, \zeta) v_2. \quad (2.38b)$$

Cross differentiation of the systems (2.37) and (2.38) shows that the eigenvalues ζ are time invariant when

$$\frac{\partial A}{\partial x} = qC - rB, \quad (2.39a)$$

$$\frac{\partial B}{\partial x} + 2i\zeta B = \frac{\partial q}{\partial t} - 2Aq, \quad (2.39b)$$

$$\frac{\partial C}{\partial x} - 2i\zeta C = \frac{\partial r}{\partial t} + 2Ar. \quad (2.39c)$$

In order that this system can be solved for A, B and C, q_t and r_t must satisfy some compatibility conditions which yield the evolution equations for q and r. AKNS solved the system (2.39) by finite expansions of A, B and C in terms of $2i\zeta$ to obtain the evolution equations for q and r including the KdV equation, the modified KdV equation, etc. Furthermore, in Ref. [110] they showed that the following nonlinear systems can be solved by means of the eigenvalue problem (2.37);

$$\begin{pmatrix} r_t \\ -q_t \end{pmatrix} + 2\Omega(L^+) \begin{pmatrix} r \\ q \end{pmatrix} = 0, \quad -\infty < x < \infty, \quad (2.40)$$

where L^+ is a certain integro-differential operator, and $\Omega(\zeta)$ is directly

related to the dispersion relation of the linearized version of (2.40) and may be permitted to be an arbitrary ratio of entire functions. They also found a class of evolution equations which can be solved exactly by means of the Schrödinger eigenvalue problem. They then presented the view that the inverse scattering method may be considered to be an extension of the ideas of the Fourier transform to nonlinear problems.

The inverse scattering method has been applied to many important equations. Zakharov and Shabat [60] applied it to the nonlinear Schrödinger equation (1.23) with negative κ , subject to the boundary condition $|u|^2 \rightarrow \text{const.}$ and $u_x \rightarrow 0$ as $x \rightarrow \pm\infty$. Zakharov [111] could find the operators L and B appropriate to the Boussinesq equation [26]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u^2}{\partial x^2} + \frac{1}{4} \frac{\partial^4 u}{\partial x^4}, \quad (2.41)$$

which describes long water waves with small but finite amplitude. However, this equation has been not yet solved by this method, though the N -soliton solution was obtained in a different method [112].

The inverse scattering method was applied by Lamb [113] and Ablowitz, Kaup and Newell [114] to the Maxwell-Bloch equation which describes the propagation of coherent optical pulses in an inhomogeneously broadened two-level medium and is reduced to (2.36b) in the simplest case [102]. Lamb dealt with the lossless propagation of a specific type of pulses (solitons) — the self-induced transparency [115], [102]. However, in general, only a certain portion of the incident pulse forms these special solitons to which the medium is transparent. The rest of energy is eventually

transferred irreversibly to the medium. This corresponds to the fact that $a(\zeta, t)$ in the relevant scattering problem varies in t , while the zeros of $a(\zeta, t)$ which correspond to the solitons do not vary in t [114].

The operators L and B for the three-wave resonant interaction were found by Zakharov and Manakov [116], which is the nonlinear interaction of the three wave packets whose central wave numbers and frequencies satisfy the triad resonance condition and occurs in various physical contexts [117]-[122]. Two types of interaction are possible — decay and explosive. The steady propagation of solitary pulses (triple solitary waves) analogous to self-induced transparency in nonlinear optics can occur in the decay type of interaction [121]-[123]. These solitary pulses behave like solitons [124],[125]. The direct and inverse scattering problem of the operator L (which is the third order differential operator) obtained in [116] were solved by Kaup [126] under the boundary condition that all of potentials in L vanish as $x \rightarrow \pm\infty$. Because of this boundary condition, his solution could not yield the above-described solitary waves.

In many applications, the sine-Gordon equation takes the form of (2.36a) in laboratory coordinates. The equation of this form was solved by Zakharov, Takhtadzhyan and Faddeev [127], Takhtadzhyan and Faddeev [128] and Kaup [129].

The inverse scattering method was applied to the two-dimensional problem, that is, the two-dimensional KdV equation by Dryuma [130] and Zakharov and Shabat [131]. This equation was first obtained by Kadomtsev and Petviashvili [132] in an intuitive way in order to investigate the two-dimensional stability of a solitary wave in a weakly dispersive medium, and was later obtained by Oikawa, Satsuma and Yajima [133] in a systematic way. Although the scattering problem of the relevant operator has been

not yet solved, the N-soliton solution has been obtained in [131] and [134].

The method was also applied to the Toda lattice which is a system of equal masses, connected by nonlinear springs subject to an exponential restoring force and was presented by Toda [135] as a solvable model. He then obtained various special solutions including solitary waves and wave trains [135] (see also [136]). The operators L and B were found by Flaschka [137] and Manakov [138] and for an infinite lattice the direct and inverse scattering problem were solved by them to find the N-soliton solution. The periodic Toda lattice was investigated by Kac and Moerbeke [139] and Date and Tanaka [140] on the basis of this method.

The KdV equation with the periodic boundary condition has been recently studied by a number of authors [141]-[148]. It was in this boundary condition that Zabusky and Kruskal [14],[28] first discovered the "soliton" through the numerical computations of the equation which were carried out in connection with the well known Fermi-Pasta-Ulam problem [149]. They observed the near recurrence of a sinusoidal initial state after a finite time and attributed this recurrence to the remarkable persistence of solitons. In the periodic case, (2.2) is often called Hill's equation and the spectrum is also invariant when the potential evolves according to the KdV equation. By solving the inverse problem of Hill's operator, the solutions of the equation were obtained by Novikov and Dubrovin [143],[144] and Its and Matveev [145] and by McKean and Moerbeke [146] for the initial data for which Hill's operator has only a finite number of gaps (forbidden bands or instability bands) in the continuous spectrum. These solutions are almost periodic in t and are the cnoidal waves for one-gap case. For two-gap case they are periodic in t , so that their initial shapes recur exactly after

a finite time, but they are shifted in position. This class of solutions contains as a degenerate limiting case the N-soliton solutions. The similar results were obtained by Lax [148] in a different way.

Other applications of the method are found in [150]-[157].

The KdV equation was cast by Gardner [158] into a Hamiltonian framework. Zakharov and Faddeev [89] showed for the KdV equation in the infinite interval that the transformation of the potential into the scattering data is a canonical transformation into action-angle variables, so that the equation is a completely integrable Hamiltonian system. They expressed the Hamiltonian, the role of which was played by the constant of motion I_3 in (2.30), in terms of the scattering data, and showed that the action variables are proportional to k_n^2 ($n=1, \dots, N$) and $\ln|a(\zeta)|^2$ and that the angle variables are proportional to $\ln(i k_n)$ ($n=1, \dots, N$) and $\arg b(\zeta)$. Many other examples of the completely integrable system have been presented [128],[138],[159]-[162].

In addition to the inverse scattering method, the methods of exact solution for nonlinear evolution equations have been developed which are convenient for finding multi-soliton solutions. One of these methods is the method based on the Bäcklund transformation which arose long ago in the study of surfaces in differential geometry [103], and is usually presented as a method for transforming one solution of an evolution equation into another solution of the same equation or into a solution of another equation. The sine-Gordon equation has long been known to admit a Bäcklund transformation [103],[102]. Since the Bäcklund transformation for the KdV equation has been recently found by Wahlquist and Estabrook [163], that for many other nonlinear equations has been obtained [164]-[168]. The Bäcklund transformation

can be used to generate multi-soliton solutions owing to the theorem of permutability due to Bianchi (see[103]) [102],[163],[169]. It can be also used as a mean for finding the eigenvalue problem associated with the evolution equation under consideration [164],[165],[170] and as that for deriving an infinite number of constants of motion of the equation [170],[171]. Miura's transformation can be, in fact, considered to be one of the Bäcklund transformations. Conversely, the Bäcklund transformations have been derived from the associated eigenvalue problems [166],[170],[172]. Although the two methods [165],[173] for finding the transformation have been developed, they are tedious and are not entirely straightforward.

Another method of exact solution has been mainly developed by Hirota (see[174]). Hirota's method is based on the transformation of dependent variable which is analogous to the Hopf-Cole transformation and reduces the original system to the system of bilinear equations. The method has been used by a number of authors to obtain N-soliton solutions of various nonlinear equations [112],[134],[175]-[179]. A method for finding the Bäcklund transformations has been developed in the framework of Hirota's method [180].

The inverse scattering method reduces a nonlinear problem to a sequence of linear problems. It permits us to solve the initial value problem unlike the other methods. It has been seen to be applicable to a wide class of nonlinear evolution equations. In particular, it was applied to the KdV equation and the nonlinear Schrödinger equation derived by the reductive perturbation method from a wide class of physical systems to reveal the interesting properties of these equations. In this method, solitons are characterized by the discrete eigenvalues, strictly speaking, the zeros of $a(\zeta)$, where $a(\zeta)$ is one of the diagonal elements of the scattering matrix.

From the viewpoint of a nonlinear analogue of the Fourier transform, the solitons are essentially nonlinear modes. The time invariance of $a(\zeta)$ permits us to derive an infinite number of the constants of motion.

It is still a highly nontrivial step to find the eigenvalue problem by means of which a given nonlinear evolution equation can be solved. However, it is not difficult to examine what nonlinear equations can be solved by means of a given eigenvalue problem. AKNS [109],[110] carried out this for the second order eigenvalue problem, i.e., the generalized Zakharov-Shabat eigenvalue problem.

Following the method in [109], we investigate in chapter 4 what nonlinear evolution equations can be solved by the inverse scattering problem of the third order eigenvalue problem. Some important examples of nonlinear evolution equations obtained thus are presented. The most interesting example of them is the system of equations describing the Langmuir waves coupled with the ion-sound waves propagating in one-direction in a one-dimensional plasma. This system of equations has solitary wave solutions — "sonic-Langmuir solitons" — which are Langmuir oscillations trapped in regions of lower density caused by the so-called ponderomotive force due to a high-frequency electromagnetic field pertaining the Langmuir oscillations. In chapter 5, the direct and inverse scattering problem of the third order differential operator associated with the system of equations are solved. The interaction properties of N sonic-Langmuir solitons are investigated. Their identities are then shown to be preserved. The formation of sonic-Langmuir solitons is studied on the basis of perturbation method. An infinite number of constants of motion are derived.

Chapter 2

Perturbation Approach to Nonlinear Systems. I.

Interaction of Solitary Waves

§ 2.1 Introduction

The present chapter is concerned with one-dimensional weakly nonlinear and weakly dispersive (or dissipative) wave systems. The reductive perturbation method can be applied such systems to yield the KdV equation (or the Burgers equation) or its modification which describes the long-time asymptotic behaviour of the single "quasi-simple" wave belonging to one family of characteristics. In particular, the KdV equation describes the unidirectional motion of shallow water waves, long waves in anharmonic lattices, ion-acoustic waves in plasma and others [12]-[24]. As described in chap. 1, through numerical and analytical studies of the KdV equation the interesting natures of the interaction of solitary waves have been revealed : when the two solitary waves approach closely, they interact, exchange their energies with one another and then separate away regaining their original wave forms. In a whole process of interaction the solitary waves are remarkably stable entities, preserving their identities through the interaction. Only their phases are altered owing to the interaction.

In one-dimensional wave systems, there exist two families (in general, n families) of characteristics which correspond to waves travelling to the right and left. The interactions of quasi-simple waves belonging to different families of characteristics cannot be investigated by using the KdV equation (or Burgers equation). The interaction of two solitary waves

travelling in opposite directions (head-on collision) was studied for the Toda lattice on the basis of the exact solution [181] or numerical computations [182]. The head-on collision of two solitary waves in shallow water waves was also studied by Benney and Luke [183] and Byatt-Smith [184]. However, in all the works the phase shifts of the solitary waves due to the head-on collision were not given explicitly.

In the present chapter, we consider the interactions of quasi-simple waves belonging to different families of characteristics by means of a generalization of the reductive perturbation method. A part of this generalization is suggested by the singular perturbation method developed by Benney and Luke in studying the interactions of two permanent shallow water waves. Another part of the generalization is to introduce the phase variables in expectation that the velocities of the two interacting waves vary in space and time owing to the mutual interactions.

The head-on collisions of two solitary waves are studied in § 2.2. For simplicity, we consider the Boussinesq equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = \frac{1}{2} \frac{\partial^2 y^2}{\partial x^2} + \epsilon \frac{\partial^4 y}{\partial x^4}, \quad (1.1)$$

where ϵ is a smallness parameter. This equation is well known in the studies of nonlinear gravity waves on shallow water layer [26] and long waves in anharmonic lattices [13],[14]. In the lowest order of the perturbation expansion, the solution for (1.1) is shown to be the sum of two solitary waves. The next order corrections come from the mutual interaction between these waves. The phase shifts in the head-on collision of two solitary waves are calculated.

In § 2.3, a more general case, that there exist n quasi-simple (not necessarily solitary) waves, is studied. The system is assumed to be governed by the equation (I.1.1), (where the notation (I.1.1) denotes the equation (1.1) in chap. 1 and the same notation is used in what follows). It is also shown that in the lowest order the solution is written as a superposition of n quasi-simple waves each of which is described by a simple nonlinear equation and the mutual interactions between them give rise to change of their phase velocities.

Two applications are given in § 2.4 ; the ion-acoustic waves in collisionless plasmas and the shallow water waves. It is important to note that although the Boussinesq equation (1.1) has been believed to approximate well the shallow water waves, it leads to an incorrect result in the head-on collision of two shallow water solitary waves.

In § 2.5, the relations of several recent papers to the present work are discussed.

§ 2.2 Head-on Collision of Solitary Waves of the Boussinesq Equation

Let us consider the Boussinesq equation (1.1). A solution corresponding to a unidirectional permanent wave can be obtained by putting $y=s(x-ct)$ and by solving

$$(1-c^2)s_{xx} + (s^2/2)_{xx} + \varepsilon s_{xxxx} = 0. \quad (2.1)$$

Imposing the boundary condition that $s=s_x=s_{xx}=s_{xxx}=0$ at $x=\infty$, we get the solitary wave solutions

$$s = H \operatorname{sech}^2 \left\{ (H/(12\varepsilon))^{1/2} (x-ct-x_0) \right\}, \quad (2.2a)$$

$$c = \pm (1 + H/3)^{1/2}. \quad (2.2b)$$

If the amplitude is of order ε , i.e., $H = \varepsilon A$, $\varepsilon \ll 1$, the solitary waves are

$$s \cong \varepsilon A \operatorname{sech}^2 \left\{ (A/12)^{1/2} (x \mp (1 + \varepsilon A/6)t) \right\}^{x_0} \quad (2.3)$$

Consider the head-on collision of two solitary waves. Of course a linear combination of two solitary waves will not satisfy (1.1) but will give rise to nonlinear interaction terms. However, since the interaction terms are of order ε^2 , it may be treated by a perturbation method.

We now introduce the following variables :

$$\xi = x - x_A - (1 + \varepsilon \lambda)t - \sum_{n=1}^{\infty} \varepsilon^n \varphi_n(x, t), \quad (2.4a)$$

$$\eta = x - x_B + (1 + \varepsilon \mu)t - \sum_{n=1}^{\infty} \varepsilon^n \psi_n(x, t), \quad (2.4b)$$

where x_A and x_B are arbitrary constants. This is a generalization of

Benny-Luke's treatment. Here, the phase variables φ_n and ψ_n are introduced in expectation that the velocities of the two solitary waves vary in space and time owing to the head-on collision. On the other hand, Benney and Luke disregarded these variables because they were mainly concerned with the stationary two-dimensional interaction of two periodic waves (cnoidal waves) travelling in different directions.

If we introduce (2.4) into (1.1) and expand y in terms of the smallness parameter ε as

$$y = \varepsilon y^{(1)} + \varepsilon^2 y^{(2)} + \dots, \quad (2.5)$$

then we get the sequence of equations corresponding to the successive powers of ε .

In the first approximation, we obtain

$$\frac{\partial^2}{\partial \xi \partial \eta} y^{(1)} = 0. \quad (2.6)$$

Equation (2.6) implies that the lowest order solution is the sum of two waves having the phases ξ and η ;

$$y^{(1)} = f(\xi) + g(\eta). \quad (2.7)$$

In the next order, we have

$$\begin{aligned} \frac{\partial^2}{\partial \xi \partial \eta} y^{(2)} = & -\frac{\partial^2}{\partial \xi \partial \eta} \left\{ \frac{1}{2} f g - \left(\varphi_1 - \frac{G}{4} \right) f' - \left(\psi_1 - \frac{F}{4} \right) g' \right\} \\ & - \frac{1}{4} \left\{ f'' + \left(\frac{f^2}{2} \right)'' - 2\lambda f'' \right\} - \frac{1}{4} \left\{ g'' + \left(\frac{g^2}{2} \right)'' - 2\mu g'' \right\}, \end{aligned} \quad (2.8)$$

$$F = \int_{-\infty}^{\xi} f(\xi') d\xi', \quad G = \int_{-\infty}^{\eta} g(\eta') d\eta' \quad (2.9)$$

where the primes denote the differentiations with respect to the arguments. Taking into account that f and g are functions of ξ and η , respectively, and requiring the boundedness of $y^{(2)}$ (non-secularity requirement for solution) we obtain

$$f''' + (f^2/2)'' - 2\lambda f'' = 0,$$

$$g''' + (g^2/2)'' - 2\mu g'' = 0.$$

These are integrated with respect to ξ and η , respectively ;

$$f'' + (f^2/2)' - 2\lambda f' = a_1, \quad (2.10a)$$

$$g'' + (g^2/2)' - 2\mu g' = a_2, \quad (2.10b)$$

where a_1 and a_2 are arbitrary constants. Suppose now that

$$\varphi_1 = G(\eta)/4, \quad (2.11a)$$

$$\psi_1 = F(\xi)/4. \quad (2.11b)$$

These determine the phase variables of the two waves. From (2.8)-(2.11), $y^{(2)}$ is given as

$$y^{(2)} = -\frac{1}{2}fg + h(\xi) + k(\eta). \quad (2.12)$$

The functions $h(\xi)$ and $k(\eta)$ are as yet arbitrary functions and are

determined by requiring the boundedness of the third order solution $y^{(3)}$.

After some lengthy but straightforward manipulations, the third order equation is written as

$$\begin{aligned}
 4 \frac{\partial^2}{\partial \xi \partial \eta} y^{(3)} = & - \frac{\partial^2}{\partial \xi^2} \{ h'' + (f - 2\lambda)h - \lambda^2 f \} - \frac{\partial^2}{\partial \eta^2} \{ k'' + (g - 2\mu)k - \mu^2 g \} \\
 & + \frac{\partial}{\partial \xi} \left[\left\{ 4 \frac{\partial \mathcal{P}_2}{\partial \eta} - k + \frac{5}{2}(\lambda + \mu)g \right\} f' \right] + \frac{\partial}{\partial \eta} \left[\left\{ 4 \frac{\partial \mathcal{Y}_2}{\partial \xi} - h + \frac{5}{2}(\lambda + \mu)f \right\} g' \right] \\
 & + \left(\frac{\partial^2}{\partial \eta^2} + \frac{11}{4} \frac{\partial^2}{\partial \xi \partial \eta} \right) \left[\left(f'' + \frac{f^2}{2} - 2\lambda f \right) g \right] + \left(\frac{\partial^2}{\partial \xi^2} + \frac{11}{4} \frac{\partial^2}{\partial \xi \partial \eta} \right) \left[\left(g'' + \frac{g^2}{2} - 2\mu g \right) f \right] \\
 & + \frac{\partial^2}{\partial \xi \partial \eta} \left[-2hg - 2kf + 6(\lambda + \mu)fg + f'g + 3f'g' + fg' \right]. \quad (2.13)
 \end{aligned}$$

The non-secularity condition of $y^{(3)}$ requires $a_1 = a_2 = 0$ in (2.10) and then

$$f'' + (f^2/2) - 2\lambda f = b_1, \quad (2.14a)$$

$$g'' + (g^2/2) - 2\mu g = b_2, \quad (2.14b)$$

where b_1 and b_2 are the integration constants. Substitution of (2.14) into (2.13) and requirement of the non-secularity of $y^{(3)}$ yield

$$\{ h'' + (f - 2\lambda)h - (\lambda^2 + b_2)f \}' = 0, \quad (2.15a)$$

$$\{ k'' + (g - 2\mu)k - (\mu^2 + b_1)g \}' = 0. \quad (2.15b)$$

Imposing that

$$\mathcal{P}_2 = \{ K(\eta) - 5(\lambda + \mu)G(\eta)/2 \}/4, \quad (2.16a)$$

$$\mathcal{Y}_2 = \{ H(\xi) - 5(\lambda + \mu)F(\xi)/2 \}/4, \quad (2.16b)$$

$$H(\xi) = \int_{-\infty}^{\xi} h(\xi') d\xi', \quad K(\eta) = \int_{-\infty}^{\eta} k(\eta') d\eta',$$

we have the solution

$$\begin{aligned} y^{(3)} = & -\frac{1}{2}(fk + gh) + \frac{3}{2}(\lambda + \mu)fg \\ & + \frac{1}{4}(f''g + 3f'g' + fg'') + \ell(\xi) + m(\eta), \end{aligned} \quad (2.17)$$

where $\ell(\xi)$ and $m(\eta)$ are arbitrary functions which will be determined in the next step.

Let us now consider the two-solitary wave problem. Setting $b_1 = b_2 = 0$ in (2.14) and imposing the boundary condition that f , g and their derivatives vanish at ξ or $\eta = \pm\infty$, we get

$$f = 6\lambda \operatorname{sech}^2((\lambda/2)^{1/2}\xi), \quad (2.18a)$$

$$g = 6\mu \operatorname{sech}^2((\mu/2)^{1/2}\eta). \quad (2.18b)$$

It is noted that these are equivalent to (2.3) with $s = \varepsilon f$, $A = 6\lambda$ and $s = \varepsilon g$, $A = 6\mu$, respectively. If we substitute (2.18a) into (2.15a) and make use of the boundary condition that h and its derivatives vanish at $\xi = \pm\infty$, then we have

$$\frac{d^2 h}{dz^2} + 4(3\operatorname{sech}^2 z - 1)h = 12\lambda^2 \operatorname{sech}^2 z, \quad (2.19)$$

where $z = (\lambda/2)^{1/2}\xi$. The solution vanishing at $\xi = \pm\infty$ can be easily obtained ;

$$h(\xi) = 3\lambda^2(\operatorname{sech}^2 z - z \operatorname{sech}^2 z \tanh z) + C \operatorname{sech}^2 z \tanh z, \quad (2.20)$$

where C is an arbitrary constant. From (2.18a), $h(\xi)$ is rewritten as

$$h(\xi) = \frac{\lambda}{2} f(\xi) + \left\{ \frac{\lambda}{4} \xi - \frac{C}{6\sqrt{2} \lambda^{3/2}} \right\} \frac{df}{d\xi}. \quad (2.20)'$$

We can say from (2.18a) and (2.20)' that up to the order ε^2 the function $\varepsilon f + \varepsilon^2 h$ agrees with

$$6\varepsilon\lambda(1+\varepsilon\lambda/2)\operatorname{sech}^2\{(\lambda(1+\varepsilon\lambda/2)/2)^{1/2}(\xi-\varepsilon\chi_A)\}, \quad (2.20)''$$

$$\chi_A = C/(6\sqrt{2}\lambda^{3/2}),$$

which is just the same as the exact solitary wave solution (2.2) with

$$C = 1 + \varepsilon\lambda \quad \text{except for the presence of the phase factor } \varepsilon\chi_A + \varepsilon\eta_1 + \varepsilon^2\eta_2.$$

The arbitrariness of the constant C can be now absorbed into that of x_A and then we can take $C=0$ without loss of generality. The function k can be obtained by replacing λ and ξ in (2.20) and (2.20)' by μ and η , respectively. By making use of (2.4), (2.5), (2.9), (2.11), (2.12) and (2.20)'', we finally obtain the solution up to the order ε^2 :

$$\begin{aligned} y = & \varepsilon A \operatorname{sech}^2((A/12)^{1/2}\xi) + \varepsilon B \operatorname{sech}^2((B/12)^{1/2}\eta) \\ & - (\varepsilon^2/2)AB \operatorname{sech}^2((A/12)^{1/2}\xi) \operatorname{sech}^2((B/12)^{1/2}\eta), \end{aligned} \quad (2.21)$$

$$A = 6\lambda(1 + \varepsilon\lambda/2), \quad B = 6\mu(1 + \varepsilon\mu/2),$$

$$\xi = x - x_A - (1 + \varepsilon\lambda)t - (\sqrt{3}/2)\varepsilon B^{1/2}(\tanh\{(B/12)^{1/2}\eta\} + 1), \quad (2.22a)$$

$$\eta = x - x_B + (1 + \varepsilon\mu)t - (\sqrt{3}/2)\varepsilon A^{1/2}(\tanh\{(A/12)^{1/2}\xi\} + 1). \quad (2.22b)$$

Now, we can estimate the phase shifts in the head-on collision of two solitary waves. Let us assume that the two solitary waves, A and B,

are at a long distance from each other at an initial instant ($t = -\infty$) ;
the solitary wave A is at $\eta = -\infty$ and $\xi = 0$ and B at $\xi = +\infty$ and $\eta = 0$.
After the collision ($t = \infty$), A is far on the right of B ; A is at $\eta = \infty$ and
 $\xi = 0$ and B at $\xi = -\infty$ and $\eta = 0$. In this case the phase shifts of A and
B, δ_A and δ_B , are given by

$$\delta_A = [\chi - (1 + \varepsilon\lambda)t]_{\xi=0, \eta=\infty} - [\chi - (1 + \varepsilon\lambda)t]_{\xi=0, \eta=-\infty} = \varepsilon(3B)^{\frac{1}{2}}, \quad (2.23a)$$

$$\delta_B = [\chi + (1 + \varepsilon\mu)t]_{\xi=-\infty, \eta=0} - [\chi + (1 + \varepsilon\mu)t]_{\xi=\infty, \eta=0} = -\varepsilon(3A)^{\frac{1}{2}}. \quad (2.23b)$$

From these, we see that after the interaction the right-going wave A is
shifted to the right of where it would have been if no interaction had
taken place and the left-going wave B is shifted to the left.

§ 2.3 Generalization to n-Wave Systems

In the previous section, we investigated the interaction of two solitary waves propagating in opposite directions. Here we consider a more general case that there exist n "quasi-simple" (not necessarily solitary) waves (simple waves under the influence of dispersion or dissipation) each of which belongs to one of n families of characteristics. For the system (I.1.1), the equation for the single quasi-simple wave belonging to one of n families of characteristics derived by Taniuti and Wei [1] by applying the reductive perturbation method. In what follows, we investigate the mutual interactions of n quasi-simple waves belonging to different n families of characteristics by generalizing the reductive perturbation method.

The system (I.1.1) is used as the basic equation. Here we shall also investigate the possibility that n quasi-simple waves are superposed to describe the wave motions in this system. The expansion of U in terms of a small parameter ε , (I.1.2) is assumed. It is also assumed that there exist the eigenvalues of $A_0(\equiv A(U=U^{(0)}))$, $\lambda_1, \lambda_2, \dots, \lambda_n$ and that they are real and distinct. We introduce the stretched variables

$$\xi_j = \varepsilon^a (x - \lambda_j t - \varepsilon^{1-a} \varphi_j(x, t)), \quad (j=1, 2, \dots, n) \quad (3.1a)$$

$$\tau = \varepsilon^{a+1} t, \quad (3.1b)$$

where $a=(p-1)^{-1} \leq 1$. We consider that U depends on x and t through the variables ξ_j ($j=1, 2, \dots, n$) and τ . These variables are the generalization of the variables (I.1.3). In (3.1), $\varphi_j(x, t)$ is introduced in anticipation that the velocities of waves vary in space and time owing to the mutual interactions. The factor ε^{1-a} comes from the following consideration :

The variation in the wave velocity due to the two-wave interaction is expected to be proportional to the product of the wave amplitude and the interaction time. The former is of order ε . The latter is considered to be the time during which the two waves pass through each other, and then estimated by dividing the width of wave ($\sim O(\varepsilon^{-a})$) with their relative velocity ($\sim O(1)$), i.e., being of order ε^{-a} . Therefore, the variation in the wave velocity is of order $\varepsilon \times \varepsilon^{-a} = \varepsilon^{1-a}$.

If we substitute (I.1.2) and (3.1) into (I.1.1) and equate the coefficients of successive powers of ε to zero, then we get a sequence of equations.

In the lowest order, we have

$$\sum_{\ell=1}^n (A_0 - \lambda_{\ell}) \frac{\partial U^{(1)}}{\partial \xi_{\ell}} = 0. \quad (3.2)$$

Let R_{ℓ} and L_{ℓ} be the right and left eigenvectors of A_0 for the eigenvalue λ_{ℓ} , respectively ;

$$A_0 R_{\ell} = \lambda_{\ell} R_{\ell}, \quad L_{\ell} A_0 = \lambda_{\ell} L_{\ell}. \quad (3.3)$$

By expanding $U^{(1)}$ with the set $\{R_j\}$ as

$$U^{(1)} = \sum_{j=1}^n f_j(\xi_1, \dots, \xi_n, \tau) R_j, \quad (3.4)$$

and by making use of the orthogonality of the eigenvectors, i.e.,

$$(L_j, R_k) = \delta_{jk}, \quad (3.5)$$

we get the equation

$$\sum_{\ell=1}^n (\lambda_j - \lambda_\ell) \frac{\partial f_j}{\partial \xi_\ell} = 0. \quad (j = 1, 2, \dots, n) \quad (3.6)$$

The general solution of (3.6) is written as

$$f_j = f_j(\gamma_1^{(j)}, \gamma_2^{(j)}, \dots, \gamma_n^{(j)}, \tau), \quad (3.7a)$$

$$\gamma_i^{(j)} = \xi_i - n_i^{(j)} \sum_{\ell=1}^n n_\ell^{(j)} \xi_\ell, \quad (3.7b)$$

$$n_i^{(j)} = (\lambda_j - \lambda_i) / \Lambda_j, \quad \Lambda_j \equiv \left[\sum_i (\lambda_j - \lambda_i)^2 \right]^{1/2}, \quad (3.7c)$$

where the vector $\gamma^{(j)}$ is the orthogonal component of the vector ξ to the n -dimensional unit vector $n^{(j)}$. Since we are interested in quasi-simple waves, we restrict ourselves to the case that f_j is a function of only one variable $\gamma_j^{(j)} (= \xi_j)$. Then, we have

$$f_j = f_j(\xi_j, \tau). \quad (3.8)$$

It should be noted here that for the case $n=2$, equation (3.6) has only the solution (3.6).

In the next order, we have

$$\begin{aligned} \sum_j (\lambda_j - \lambda_\ell) \frac{\partial g_\ell}{\partial \xi_j} &= \sum_{j \neq \ell} \sum_m (L_\ell, (R_m \nabla) A_0 R_j) f_m \frac{\partial f_j}{\partial \xi_j} \\ &+ \sum_{j \neq \ell} (L_\ell, \sum_{\beta=1}^S \prod_{\alpha=1}^P (K_{\alpha 0}^\beta - \lambda_j H_{\alpha 0}^\beta) R_j) \frac{\partial^P f_j}{\partial \xi_j^P} \\ &+ \frac{\partial f_\ell}{\partial \tau} + (L_\ell, (R_\ell \nabla) A_0 R_\ell) f_\ell \frac{\partial f_\ell}{\partial \xi_\ell} \end{aligned}$$

$$\begin{aligned}
& + (L_\ell, \sum_{\beta=1}^S \prod_{\alpha=1}^p (K_{\alpha 0}^\beta - \lambda_\ell H_{\alpha 0}^\beta) R_\ell) \frac{\partial^p f_\ell}{\partial \xi_\ell^p} \\
& + \sum_{j \neq \ell} \left\{ (\lambda_j - \lambda_\ell) \frac{\partial \varphi_\ell}{\partial \xi_j} + (L_\ell, (R_j \nabla) A_0 R_\ell) f_j \right\} \frac{\partial f_\ell}{\partial \xi_\ell}, \quad (3.9)
\end{aligned}$$

where the $g_\ell^{\prime s}$ are the coefficients of the expansion of $U^{(2)}$ with $\{R_\ell\}$, i.e.,
 $U^{(2)} = \sum_{\ell=1}^n g_\ell(\xi_1, \xi_2, \dots, \xi_n, \tau) R_\ell.$

Now suppose that the variables $g_\ell^{\prime s}$ satisfy

$$\sum_{j \neq \ell} (\lambda_j - \lambda_\ell) \frac{\partial \varphi_\ell}{\partial \xi_j} = - \sum_{j \neq \ell} (L_\ell, (R_j \nabla) A_0 R_\ell) f_j, \quad (3.10)$$

i.e.,

$$\varphi_\ell = \sum_{j \neq \ell} (\lambda_\ell - \lambda_j)^{-1} (L_\ell, (R_j \nabla) A_0 R_\ell) \int_{\xi_j}^{\xi_j} f_j(\xi) d\xi + \theta_\ell(\gamma_1^{(\ell)}, \dots, \gamma_n^{(\ell)}, \tau), \quad (3.11)$$

where θ_ℓ is determined by the boundary conditions for φ_ℓ . From the non-secularity conditions for $g_\ell^{\prime s}$, we finally obtain the equations for quasi-simple waves (see Appendix) :

$$\frac{\partial f_\ell}{\partial \tau} + \alpha_\ell f_\ell \frac{\partial f_\ell}{\partial \xi_\ell} + \beta_\ell \frac{\partial^p f_\ell}{\partial \xi_\ell^p} = 0, \quad (3.12a)$$

$$\alpha_\ell = (L_\ell, (R_\ell \nabla) A_0 R_\ell), \quad (3.12b)$$

$$\beta_\ell = (L_\ell, \sum_{\beta=1}^S \prod_{\alpha=1}^p (K_{\alpha 0}^\beta - \lambda_\ell H_{\alpha 0}^\beta) R_\ell). \quad (3.12c)$$

The n-quasi-simple wave systems are governed by the equations (3.12) and (3.11) with (3.1). Each quasi-simple waves are described by the simple nonlinear equation (3.12). The equation becomes, for a special value of the parameter p , the Burgers equation ($p=2$) and the KdV equation ($p=3$)

which are exactly solvable. In the lowest order approximation, the wave motions are described by the superpositions of these quasi-simple waves and the mutual interactions between them are included in the phase variables \mathcal{H}'_s .

§ 2.4 Applications

Let us consider the case that the quasi-simple waves are described by their respective KdV equation, i.e., the case $p=3$. In this case, it is particularly important to investigate the mutual interactions of solitary waves because they play an fundamental role in the solutions of their respective KdV equations. Here two examples of such interactions are given.

A) Ion acoustic waves

For a collisionless plasma of cold ions and warm electrons, the basic normalized system of equations is as follows [1],[17] :

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\left(\frac{1}{n}\frac{\partial n}{\partial x}\right) = 0, \quad (4.1a)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{n}\frac{\partial n}{\partial x} = 0, \quad (4.1b)$$

where n is the electron number density and u the ion velocity. In (4.1), the quantities are normalized in terms of the following characteristic quantities : the mean density n_0 , the sound velocity of the ion acoustic wave $(T_e/m_i)^{1/2}$ and the Debye length $(T_e/4\pi n_0 e^2)^{1/2}$, where T_e is the constant electron temperature, m_i the ion mass and e the electric charge of an electron.

By comparing (4.1) with (I.1.1), we have

$$U = \begin{pmatrix} n \\ u \end{pmatrix}, \quad A = \begin{pmatrix} u & n \\ n^{-1} & u \end{pmatrix}, \quad H'_1 = 0, \quad H'_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$H'_3 = 0, \quad K'_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K'_2 = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}, \quad K'_3 = \begin{pmatrix} \bar{n}^{-1} & 0 \\ 0 & 0 \end{pmatrix},$$

$$p=3, \quad s=1.$$

For the case of $U^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have

$$A_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\lambda_1 = 1, \quad R_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad L_1 = \left(\frac{1}{2}, \frac{1}{2}\right),$$

$$\lambda_2 = -1, \quad R_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad L_2 = \left(\frac{1}{2}, -\frac{1}{2}\right),$$

$$(R_1 \nabla) A_0 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (R_2 \nabla) A_0 = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix},$$

$$\sum_{\beta=1}^S \prod_{\alpha=1}^p (K_{\alpha 0}^{\beta} - \lambda_j H_{\alpha 0}^{\beta}) = \lambda_j \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Substitution of these into (3.11) and (3.12) yields

$$\frac{\partial f_1}{\partial \tau} + f_1 \frac{\partial f_1}{\partial \xi_1} + \frac{1}{2} \frac{\partial^3 f_1}{\partial \xi_1^3} = 0, \quad \frac{\partial f_2}{\partial \tau} - f_2 \frac{\partial f_2}{\partial \xi_2} - \frac{1}{2} \frac{\partial^3 f_2}{\partial \xi_2^3} = 0, \quad (4.2a, b)$$

$$\xi_1 = \varepsilon^{1/2} \left\{ \chi - t + \frac{\varepsilon^{1/2}}{2} \int^{\xi_2} f_2(\xi) d\xi \right\}, \quad (4.2c)$$

$$\xi_2 = \varepsilon^{1/2} \left\{ \chi + t + \frac{\varepsilon^{1/2}}{2} \int^{\xi_1} f_1(\xi) d\xi \right\}, \quad (4.2d)$$

$$\tau = \varepsilon^{3/2} t, \quad (4.2e)$$

where $n = 1 + \varepsilon n_1 + \dots$, $u = \varepsilon u_1 + \dots$, $f_1 = (n_1 + u_1)/2$ and $f_2 = (n_1 - u_1)/2$.

For the head-on collision of two solitary waves, we get

$$f_1 = A \operatorname{sech}^2 \left\{ \left(\frac{A}{6}\right)^{1/2} \left(\xi_1 - \frac{A}{3} \tau\right) \right\}, \quad f_2 = B \operatorname{sech}^2 \left\{ \left(\frac{B}{6}\right)^{1/2} \left(\xi_2 + \frac{B}{3} \tau\right) \right\}, \quad (4.3a, b)$$

$$\xi_1 = \varepsilon^{1/2} [\chi - t + (\frac{3}{2}\varepsilon B)^{1/2} \tanh \{ (\frac{B}{\delta})^{1/2} (\xi_2 + \frac{B}{3}\tau) \} + \text{const}], \quad (4.3c)$$

$$\xi_2 = \varepsilon^{1/2} [\chi + t + (\frac{3}{2}\varepsilon A)^{1/2} \tanh \{ (\frac{A}{\delta})^{1/2} (\xi_1 - \frac{A}{3}\tau) \} + \text{const}]. \quad (4.3d)$$

The phase shifts of the two solitary due to the head-on collision, δ_A and δ_B , can be estimated as

$$\delta_A = [\chi - t]_{\xi_1=0, \xi_2=\infty} - [\chi - t]_{\xi_1=0, \xi_2=-\infty} = -(6\varepsilon B)^{1/2}, \quad (4.4a)$$

$$\delta_B = [\chi + t]_{\xi_1=-\infty, \xi_2=0} - [\chi + t]_{\xi_1=\infty, \xi_2=0} = (6\varepsilon A)^{1/2}. \quad (4.4b)$$

These show that the right-going wave f_1 is shifted to the left of where it would have been if no interaction had taken place and the left-going wave f_2 to the right.

B) Shallow water waves

The basic system of equations used here is the following system :

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (hu) = 0, \quad (4.5a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} + \frac{1}{3} \frac{\partial^3 h}{\partial t^2 \partial x} = 0, \quad (4.5b)$$

where h is the water depth normalized by the undisturbed depth h_0 and u the mean water velocity normalized by the characteristic velocity $(gh_0)^{1/2}$, where g is the acceleration due to gravity. These equations are those which Whitham [11],[54] has referred to as the Boussinesq equations. The Boussinesq equation (1.1) can be derived from (4.5) by eliminating

u and by using some approximations. In this case, U, A, etc. are given as follows :

$$U = \begin{pmatrix} h \\ u \end{pmatrix}, \quad A = \begin{pmatrix} u & h \\ 1 & u \end{pmatrix}, \quad H'_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \end{pmatrix}, \quad H'_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$H'_3 = 0, \quad K'_1 = 0, \quad K'_2 = 0, \quad K'_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$p=3, \quad s=1.$$

For the constant state, $U^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we have the same eigenvalues and the same eigenvectors as those in the previous example A). After elementary calculations, we obtain

$$\frac{\partial f_1}{\partial \tau} + \frac{3}{2} f_1 \frac{\partial f_1}{\partial \xi_1} + \frac{1}{6} \frac{\partial^3 f_1}{\partial \xi_1^3} = 0, \quad \frac{\partial f_2}{\partial \tau} - \frac{3}{2} f_2 \frac{\partial f_2}{\partial \xi_2} - \frac{1}{6} \frac{\partial^3 f_2}{\partial \xi_2^3} = 0, \quad (4.6a,b)$$

$$\xi_1 = \varepsilon^{1/2} \left\{ \chi - \tau + \frac{1}{4} \varepsilon^{1/2} \int^{\xi_2} f_2(\xi) d\xi \right\}, \quad (4.6c)$$

$$\xi_2 = \varepsilon^{1/2} \left\{ \chi + \tau + \frac{1}{4} \varepsilon^{1/2} \int^{\xi_1} f_1(\xi) d\xi \right\}, \quad (4.6d)$$

$$\tau = \varepsilon^{3/2} t, \quad (4.6e)$$

where $h = 1 + \varepsilon h_1 + \dots$, $u = \varepsilon u_1 + \dots$, $f_1 = (h_1 + u_1)/2$ and $f_2 = (h_1 - u_1)/2$.

For the head-on collision of the two solitary waves,

$$f_1 = A \operatorname{sech}^2 \left\{ \left(\frac{3}{4} A \right)^{1/2} \left(\xi_1 - \frac{A}{2} \tau \right) \right\}, \quad f_2 = B \operatorname{sech}^2 \left\{ \left(\frac{3}{4} B \right)^{1/2} \left(\xi_2 + \frac{B}{2} \tau \right) \right\}, \quad (4.7a,b)$$

the phase shifts are given by

$$\delta_A = -\left(\varepsilon \frac{B}{3}\right)^{1/2}, \quad \delta_B = \left(\varepsilon \frac{A}{3}\right)^{1/2}. \quad (4.8a,b)$$

In this case also, the right-going wave is shifted to the left of where it would have been if no interaction had taken place and the left-going wave to the right.

It must be noted that the signs of these phase shifts differ from those of the phase shifts in the Boussinesq equation (1.1). Therefore, (1.1) is inapplicable to the studies of the mutual interactions of two finite amplitude shallow water waves travelling in opposite directions. This has been also pointed out by Long [185] and Byatt-Smith [184]. This is due to an incorrect approximation in the derivation of (1.1).

§ 2.5 Concluding Remarks

In the present chapter, we have presented a new approximation method, which is a generalization of the reductive perturbation method, for studying wave motions in weakly nonlinear and weakly dispersive or dissipative systems, in particular, the mutual interactions of "quasi-simple waves" belonging to different families of characteristics in such systems. The head-on collision of two solitary waves of the Boussinesq equation (1.1) has been first investigated by introducing the phase variables and in particular, the phase shifts of the solitary waves due to the collision have been found. Next, a more general case, i.e., n -quasi-simple wave system has been considered. In the lowest order approximation, the wave motion can be described by the superposition of the n quasi-simple waves, which are governed by the simple nonlinear equations (3.12). The mutual interactions between these quasi-simple waves are included in the phase variables. The present approximation method has been applied to the two ion-acoustic solitary waves and to that of two shallow water solitary waves. For both cases, the phase shifts of solitary waves have been calculated.

The exact solution of the Boussinesq equation (1.1) describing the head-on collision of two solitary waves was found by Toda and Wadati [186] and Hirota [112]. The comparison between this exact solution and the results obtained in § 2.2 justifies the method developed there.

The generalized reductive perturbation method developed in § 2.3 was applied by Tatsumi and Tokunaga [187] to a compressible, viscous and heat-conducting fluid to investigate the interactions of weak nonlinear disturbances including shocks, expansion waves and contact surfaces.

According to them, the nonlinear waves belonging to different families of characteristics behave almost independently of each other, while those belonging to the same family are governed either by the Burgers equation or by the heat conduction equation. They applied the result to one-dimensional shock turbulence in a compressible fluid and found that the law of energy decay of shock turbulence is identical to that of the Burgers turbulence. Ogino and Takeda [188] carried out the computer simulation for ion-acoustic waves on the basis of a fluid model. It is reported in [188] that their result can be well described by the superposition of the quasi-simple waves propagating in opposite directions governed by the equations (4.2) and the phase shifts in the head-on collision of two solitary waves agree well with (4.4). They also showed the following : Though the Boussinesq equation (1.1) can be derived from the fluid model by means of a certain approximation, the equation predicts incorrect results for the head-on collision of solitary waves.

Very recently, Maxworthy [189] has investigated experimentally the head-on collision between two solitary waves with equal amplitudes in shallow water. The experiments show that the maximum amplitude attained by the wave during the interaction is always greater than the sum of the amplitudes of two solitary waves and that the phase shifts due to the interaction have the same signs with (4.8). In this respect, the theoretical results based on the Boussinesq equations (4.5) agree with the experimental results (By applying the method developed in § 2.2 to (4.5), we can easily obtain the result for the maximum amplitude in agreement with the experiments). However, the values of phase shifts measured in the experiments are considerably greater than the values given by (4.8), and the former appears

to be independent of the amplitude, while the latter is proportional to the square root of the amplitude as seen from (4.8). In the experiments, waves of very small amplitude could not be measured and collapse of wave at the wave peak in the collision was observed. Therefore, the experimental results may not be possible to compare the results in B) of § 2.4.

However, it is an interesting problem to investigate why the discrepancy between the experimental results and the theoretical predictions occurs.

Appendix

Derivation of (3.12).

Let y_ℓ satisfy the following equation :

$$\sum_{j=1}^n (\lambda_\ell - \lambda_j) \frac{\partial y_\ell}{\partial \xi_j} = F(\xi_1, \dots, \xi_n), \quad (\text{A.1})$$

where $\lambda_1, \dots, \lambda_n$ are real and distinct numbers. Define the unit vector $n_j^{(\ell)}$ ($j=1, \dots, n$) as

$$n_j^{(\ell)} = (\lambda_\ell - \lambda_j) / \Lambda_\ell, \quad (\text{A.2})$$

$$\Lambda_\ell = \left[\sum_{j=1}^n (\lambda_\ell - \lambda_j)^2 \right]^{1/2}. \quad (\text{A.3})$$

Equation (A.1) then takes the form

$$\sum_{j=1}^n n_j^{(\ell)} \frac{\partial y_\ell}{\partial \xi_j} = \Lambda_\ell^{-1} F(\xi_1, \dots, \xi_n). \quad (\text{A.1}')$$

The solution of (A.1') is easily obtained ;

$$\begin{aligned} y_\ell &= \Lambda_\ell^{-1} \int ds' F(\gamma_1^{(\ell)} + n_1^{(\ell)} s', \gamma_2^{(\ell)} + n_2^{(\ell)} s', \dots, \gamma_n^{(\ell)} + n_n^{(\ell)} s') \\ &\quad + h_\ell(\gamma_1^{(\ell)}, \gamma_2^{(\ell)}, \dots, \gamma_n^{(\ell)}), \end{aligned} \quad (\text{A.4})$$

$$s = \sum_{j=1}^n n_j^{(\ell)} \xi_j, \quad \gamma_j^{(\ell)} = \xi_j - n_j^{(\ell)} \sum_{k=1}^n n_k^{(\ell)} \xi_k,$$

where h_ℓ is the homogeneous solution for (A.1'). From (A.2), we have always $n_\ell^{(\ell)} = 0$ and $\gamma_\ell^{(\ell)} = \xi_\ell$.

With the aid of (A.4) the solution for (3.9) is obtained :

$$\begin{aligned}
g_\ell = & - \sum_{\substack{j \\ (j \neq m, j \neq \ell, m \neq \ell)}} \sum_m \Lambda_\ell^{-1} (L_\ell, (R_m \nabla) A_0 R_j) \int_0^s ds' f_m(\gamma_m^{(\ell)} + \eta_m^{(\ell)} s') \frac{\partial}{\partial \eta_j^{(\ell)}} f_j(\gamma_j^{(\ell)} + \eta_j^{(\ell)} s') \\
& - \sum_{j \neq \ell} S_{\ell j}(\xi_j, \xi_\ell) - \Lambda_\ell^{-1} T_\ell(\xi_\ell) s + h_\ell(\gamma_1^{(\ell)}, \gamma_2^{(\ell)}, \dots, \gamma_n^{(\ell)}), \quad (A.5)
\end{aligned}$$

$$\begin{aligned}
S_{\ell j} = & (\lambda_\ell - \lambda_j)^{-1} \{ (L_\ell, (R_j \nabla) A_0 R_j) \frac{f_\ell^2}{2} + (L_\ell, (R_j \nabla) A_0 R_j) f_\ell f_j \\
& + (L_\ell, \sum_{\beta=1}^s \prod_{\alpha=1}^p (K_{\alpha 0}^\beta - \lambda_j H_{\alpha 0}^\beta) R_j) \frac{\partial^{p-1} f_j}{\partial \xi_j^{p-1}} \},
\end{aligned}$$

$$T_\ell = \frac{\partial f_\ell}{\partial \tau} + \alpha_\ell f_\ell \frac{\partial f_\ell}{\partial \xi_\ell} + \beta_\ell \frac{\partial^p f_\ell}{\partial \xi_\ell^p}.$$

If we require the boundedness of g_ℓ in (A.5) (the non-secularity condition of g_ℓ), the term proportional to s in (A.5) must vanish, i.e.,

$$T_\ell = 0. \quad (A.6)$$

This is nothing but the equation (3.12a).

Chapter 3

Perturbation Approach to Nonlinear Systems. II.

Interaction of Nonlinear Modulated Waves

§ 3.1 Introduction

In the previous chapter, we investigated the interaction of weakly nonlinear acoustic waves in weakly dispersive (or dissipative) systems, that is, that of "quasi-simple" waves belonging to different families of characteristics in such systems. The reductive perturbation method for long waves [1] was extended to show that in the lowest order approximation, such nonlinear systems can be described by the superpositions of the quasi-simple waves which are governed by their respective Korteweg-de Vries (or Burgers) equations.

In the present chapter, we present a similar extension of the reductive perturbation method for strongly dispersive systems [41] to the case that plane waves interact each other in such systems. In the system in which many waves coexist, nonlinear interactions consist of two parts, the self- and mutual-interactions. Even if the other waves do not exist, the self-interaction still remains, giving rise to the nonlinear modulation of the plane wave. In a weakly nonlinear system, the self-modulated wave is described by the nonlinear Schrödinger equation (I.1.16) [41]. Here we investigated the mutual-interaction of the self-modulated waves and in order to avoid complexity due to the resonance coupling between them, we restrict ourselves to a simple system consisting of only two interacting self-modulated waves. Furthermore, the self-modulated plane waves are assumed to be localized wave packets. When the differences in

the carrier wave numbers and in the carrier wave frequencies of them are of order ε , where ε is a small parameter denoting the order of magnitude of the wave amplitudes, the effect of mutual-interaction can be included in the self-modulation phenomena, that is, it can be found by solving the nonlinear Schrödinger equation. On the other hand, if they are of the order of unity, the mutual-interaction affects the velocities and the carrier wave frequencies of the self-modulated wave packet. In order to show this, we introduce the stretched coordinates and the renormalized quantities, i.e., the velocity changes and the frequency shifts of the wave packets. It is shown that in the lowest order, such a two-wave system is approximated by the superposition of the two self-modulated waves which are governed by their respective nonlinear Schrödinger equations.

In § 3.2, the perturbation method to treat the interaction between two modulated waves is established for the Klein-Gordon equation with the cubic interaction [190]. In § 3.3, the interaction between two envelope solitary waves (solitons) is studied. The method is generalized to the general system (I.1.13) in § 3.4.

§ 3.2 Method of Solution

Here we deal with the nonlinear Klein-Gordon equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + m^2 y - \beta y^3 = 0, \quad (2.1)$$

where m and β are real constants. Neglecting the nonlinear term in (2.1), we obtain the dispersion relation $\omega^2 = k^2 + m^2$, where k and ω are the wave number and the frequency of a plane wave. When the reductive perturbation method is applied to (2.1), the following nonlinear Schrödinger equation is obtained for the slowly varying complex amplitude $\mathcal{P}(\xi, \tau)$ of the modulated wave,

$$i \frac{\partial \mathcal{P}}{\partial \tau} + \frac{m^2}{2\omega^3} \frac{\partial^2 \mathcal{P}}{\partial \xi^2} + \frac{3\beta}{2\omega} |\mathcal{P}|^2 \mathcal{P} = 0, \quad (2.2)$$

where ξ and τ are the stretched coordinates given by the equations (I.1.15) [48].

We consider the interaction between two modulated plane waves.

Let k_1, ω_1 and k_2, ω_2 satisfy the dispersion relation,

$$\omega_i^2 = k_i^2 + m^2. \quad (i = 1, 2) \quad (2.3)$$

We now introduce the following stretched variables :

$$\xi_i = \varepsilon \{ x - v_i t - \sum_{j=0}^{\infty} \varepsilon^j \psi_i^{(j)}(\xi_1, \xi_2, \tau) \}, \quad \tau = \varepsilon^2 t, \quad (i=1, 2) \quad (2.4)$$

where ε is a small parameter, λ_1 and λ_2 are the group velocities, that is,

$$\lambda_i \equiv \left(\frac{\partial \omega}{\partial k} \right)_{k=k_i} = \frac{k_i}{\omega_i}, \quad (i=1, 2) \quad (2.5)$$

and $|\lambda_1 - \lambda_2|$ is assumed to be of the order of unity. The form of y is then anticipated as

$$y = \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} \sum_{\ell, n=-\infty}^{\infty} y_{\ell, n}^{(\alpha)}(\xi_1, \xi_2, \tau) Z_{\ell, n}, \quad (2.6a)$$

$$Z_{\ell, n} = \exp \left[i\ell \{k_1 x - \omega_1 t + \sum_{j=1}^{\infty} \varepsilon^j \Omega_1^{(j)}(\xi_1, \xi_2, \tau)\} + i n \{k_2 x - \omega_2 t + \sum_{j=1}^{\infty} \varepsilon^j \Omega_2^{(j)}(\xi_1, \xi_2, \tau)\} \right]. \quad (2.6b)$$

We now suppose $\Omega_i^{(r)}$ and $y_i^{(r)}$ ($i=1,2$) to be real. It is then required from reality of y that

$$y_{-\ell, -n}^{(\alpha)} = y_{\ell, n}^{(\alpha)*}. \quad (2.7)$$

The stretched coordinates (2.4) and the expansion form (2.6) are the same as (I.1.15) and (I.1.14), respectively, but the variables $y_i^{(r)}$ and $\Omega_i^{(r)}$ ($i=1,2$) are introduced by taking into account that the velocities and the frequencies of the two wave packets vary in space and time owing to the mutual-interaction between them.

Substituting (2.4) and (2.6) into (2.1) and equating the coefficients of various powers of ε in the same harmonics to zero, we get the sequence of equations to be solved :

$$O(\varepsilon): W_{\ell, n} y_{\ell, n}^{(1)} = 0, \quad (2.8)$$

$$O(\varepsilon^2): W_{\ell, n} y_{\ell, n}^{(2)} + 2i \mathcal{L}_{\ell, n} [y_{\ell, n}^{(1)}] = 0, \quad (2.9)$$

$$O(\varepsilon^3): W_{\ell, n} y_{\ell, n}^{(3)} + 2i \mathcal{L}_{\ell, n} [y_{\ell, n}^{(2)}] - 2i(\ell\omega_1 + n\omega_2) \frac{\partial y_{\ell, n}^{(1)}}{\partial \tau} + \left(\lambda_1 \frac{\partial}{\partial \xi_1} + \lambda_2 \frac{\partial}{\partial \xi_2} \right)^2 y_{\ell, n}^{(1)} - \left(\frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right)^2 y_{\ell, n}^{(1)} - 2y_{\ell, n}^{(1)} \mathcal{L}_{\ell, n} [\ell\Omega_1^{(1)} + n\Omega_2^{(1)}]$$

$$\begin{aligned}
& -2i\mathcal{L}_{\ell,n}[\psi_1^{(0)}]\frac{\partial y_{\ell,n}^{(1)}}{\partial \xi_1} - 2i\mathcal{L}_{\ell,n}[\psi_2^{(0)}]\frac{\partial y_{\ell,n}^{(1)}}{\partial \xi_2} \\
& -\beta \sum_{\ell',n'} \sum_{\ell'',n''} y_{\ell',n'}^{(1)} y_{\ell'',n''}^{(1)} y_{\ell-\ell'-\ell'', n-n'-n''}^{(1)} = 0,
\end{aligned} \tag{2.10}$$

and so on, where

$$W_{\ell,n} = -(\ell\omega_1 + n\omega_2)^2 + (\ell k_1 + n k_2)^2 + m^2, \tag{2.11}$$

$$\mathcal{L}_{\ell,n} = n\omega_2(\lambda_1 - \lambda_2)\frac{\partial}{\partial \xi_1} + \ell\omega_1(\lambda_2 - \lambda_1)\frac{\partial}{\partial \xi_2}. \tag{2.12}$$

From (2.3) and (2.11), we obtain

$$W_{\ell,n} = 0, \quad \text{for } |\ell| + |n| = 1. \tag{2.13}$$

Here we assume that

$$W_{\ell,n} \neq 0, \quad \text{for } |\ell| + |n| \neq 1. \tag{2.14}$$

It should be noted that (2.14) cannot be satisfied for a special combination of k_1 , k_2 and m . Then, the equation (2.8) yields

$$y_{\ell,n}^{(1)} = 0, \quad \text{for } |\ell| + |n| \neq 1. \tag{2.15}$$

By making use of (2.13) and (2.15), we obtain from (2.9) the following :

$$y_{\ell,n}^{(2)} = 0, \quad \text{for } |\ell| + |n| \neq 1. \tag{2.16}$$

In view of (2.12) and (2.13), (2.9) for $\ell = 1$, $n=0$ and for $\ell = 0$, $n=1$ take the forms

$$\frac{\partial y_{1,0}^{(1)}}{\partial \xi_2} = 0 \quad \text{and} \quad \frac{\partial y_{0,1}^{(1)}}{\partial \xi_1} = 0,$$

respectively. Thus, we get

$$y_{1,0}^{(1)} = f_1(\xi_1, \tau), \quad y_{0,1}^{(1)} = f_2(\xi_2, \tau) \quad (2.17)$$

The functions f_1 and f_2 will be determined by the non-secularity condition in the next order equation for $y_{\ell,n}^{(2)}$.

In the next order, we have, for $\ell=1$, $n=0$,

$$\begin{aligned} & i(\lambda_2 - \lambda_1) \frac{\partial y_{1,0}^{(2)}}{\partial \xi_2} - \left(i \frac{\partial f_1}{\partial \tau} + \frac{m^2}{2\omega_1^3} \frac{\partial^2 f_1}{\partial \xi_1^2} + \frac{3\beta}{2\omega_1} |f_1|^2 f_1 \right) \\ & + f_1 \left\{ (\lambda_1 - \lambda_2) \frac{\partial \Omega_1^{(1)}}{\partial \xi_2} - \frac{3\beta}{\omega_1} |f_2|^2 \right\} - i(\lambda_2 - \lambda_1) \frac{\partial \psi_1^{(0)}}{\partial \xi_2} \frac{\partial f_1}{\partial \xi_1} = 0. \end{aligned} \quad (2.18)$$

We now suppose that $\Omega_1^{(1)}$ and $\psi_1^{(0)}$ satisfy

$$\frac{\partial \Omega_1^{(1)}}{\partial \xi_2} = \frac{3\beta}{\omega_1(\lambda_1 - \lambda_2)} |f_2|^2, \quad \text{and} \quad \frac{\partial \psi_1^{(0)}}{\partial \xi_2} = 0,$$

respectively, that is,

$$\Omega_1^{(1)} = \tilde{\Phi}_1(\xi_2, \tau) + \Phi_1(\xi_1, \tau), \quad (2.19)$$

$$\tilde{\Phi}_1(\xi_2, \tau) = \frac{3\beta}{\omega_1(\lambda_1 - \lambda_2)} \int_0^{\xi_2} |f_2(\xi, \tau)|^2 d\xi, \quad (2.20)$$

$$\psi_1^{(0)} = \psi_1(\xi_1, \tau).$$

Again, the functions $\Phi_1(\xi_1, \tau)$ and $\Psi_1(\xi_1, \tau)$ are due to the self-interaction and determined from the next order equations. The non-secularity condition for $y_{1,0}^{(2)}$ (note that f_1 does not depend on ξ_2) immediately yields

$$i \frac{\partial f_1}{\partial \tau} + \frac{m^2}{2\omega_1^3} \frac{\partial^2 f_1}{\partial \xi_1^2} + \frac{3\beta}{2\omega_1} |f_1|^2 f_1 = 0 \quad (2.21)$$

$$\frac{\partial y_{1,0}^{(2)}}{\partial \xi_2} = 0, \quad \text{i.e., } y_{1,0}^{(2)} = g_1(\xi_1, \tau), \quad (2.22)$$

where $g_1(\xi_1, \tau)$ is yet to be determined. In order to find g_1 , we must use the next order equation (coefficient of ε^4).

For $\ell = 0$, $n=1$, the results can be obtained by replacing the subscripts 1 and 2 with 2 and 1, respectively.

The equation (2.21) is identical with (2.2) which describes the wave modulation due to the nonlinear self-interaction. On the other hand, $\tilde{\Phi}_1(\xi_2, \tau)$ and $\tilde{\Phi}_2(\xi_1, \tau)$ describe the effects due to the mutual-interaction the two waves, giving rise to the variations of wave numbers and frequencies.

It is readily found from (2.14), (2.15) and (2.16) that the equation (2.10) for $|\ell| + |n| \neq 1$ gives the equations

$$y_{\ell,n}^{(3)} = 0, \quad \text{for } |\ell| + |n| \neq 1, |\ell| + |n| \neq 3,$$

$$y_{3,0}^{(3)} = \frac{\beta}{W_{3,0}} f_1^3, \quad y_{0,3}^{(3)} = \frac{\beta}{W_{0,3}} f_2^3, \quad y_{2,1}^{(3)} = \frac{3\beta}{W_{2,1}} f_1^2 f_2,$$

$$y_{1,2}^{(3)} = \frac{3\beta}{W_{1,2}} f_1 f_2^2, \quad y_{2,-1}^{(3)} = \frac{3\beta}{W_{2,-1}} f_1^2 f_2^*, \quad \dots$$

The equation corresponding to the fourth order of ε becomes

lengthier. Therefore, only the result for $\ell = 1$, $n=0$ is written down :

$$\begin{aligned}
 & i(\lambda_2 - \lambda_1) \frac{\partial}{\partial \xi_2} \left\{ \psi_{1,0}^{(3)} - \frac{m^2}{2\omega_1 \omega_2^2 (\lambda_2 - \lambda_1)} f_1 \frac{\partial \tilde{\Phi}_1}{\partial \xi_2} \right\} - G_1(\xi_1, \tau) \\
 & - i(\lambda_2 - \lambda_1) \frac{\partial f_1}{\partial \xi_1} \left\{ \frac{\partial \psi_1^{(1)}}{\partial \xi_2} - \frac{1 - \lambda_1 \lambda_2}{\omega_1 (\lambda_1 - \lambda_2)} \frac{\partial \tilde{\Phi}_1}{\partial \xi_2} \right\} \\
 & - (\lambda_2 - \lambda_1) f_1 \left\{ \frac{\partial \Omega_1^{(2)}}{\partial \xi_2} - H_1(\xi_2, \tau) \right\} = 0, \tag{2.23}
 \end{aligned}$$

where

$$\begin{aligned}
 G_1(\xi_1, \tau) = & i \frac{\partial g_1}{\partial \tau} + \frac{m^2}{2\omega_1^3} \frac{\partial^2 g_1}{\partial \xi_1^2} + \frac{3\beta}{2\omega_1} |f_1|^2 g_1 + \frac{3\beta}{2\omega_1} f_1 (f_1^* g_1 + f_1 g_1^*) \\
 & + \frac{\lambda_1}{\omega_1} \frac{\partial^2 f_1}{\partial \xi_1 \partial \tau} - f_1 \frac{\partial \Phi_1}{\partial \tau} + i \frac{m^2}{2\omega_1^3} f_1 \frac{\partial^2 \tilde{\Phi}_1}{\partial \xi_1^2} + i \frac{m^2}{\omega_1^3} \frac{\partial f_1}{\partial \xi_1} \frac{\partial \tilde{\Phi}_1}{\partial \xi_1} \\
 & - i \frac{\partial f_1}{\partial \xi_1} \frac{\partial \psi_1}{\partial \tau} - \frac{m^2}{2\omega_1^3} \frac{\partial f_1}{\partial \xi_1} \frac{\partial^2 \psi_1}{\partial \xi_1^2} - \frac{m^2}{\omega_1^3} \frac{\partial^2 f_1}{\partial \xi_1^2} \frac{\partial \psi_1}{\partial \xi_1}, \tag{2.24}
 \end{aligned}$$

$$H_1(\xi_2, \tau) = \frac{\partial \psi_2}{\partial \xi_2} \frac{\partial \tilde{\Phi}_1}{\partial \xi_2} + \frac{1}{(\lambda_2 - \lambda_1)} \frac{\partial \tilde{\Phi}_1}{\partial \tau} + \frac{3\beta}{\omega_1 (\lambda_1 - \lambda_2)} (f_2^* g_2 + f_2 g_2^*). \tag{2.25}$$

Following the previous step, let us take $\psi_i^{(1)}$ and $\Omega_i^{(2)}$ in (2.23) so as to satisfy the relations

$$\frac{\partial \psi_1^{(1)}}{\partial \xi_2} = \frac{1 - \lambda_1 \lambda_2}{\omega_1 (\lambda_1 - \lambda_2)} \frac{\partial \tilde{\Phi}_1}{\partial \xi_2}, \tag{2.26}$$

and

$$\frac{\partial \Omega_1^{(2)}}{\partial \xi_2} = H_1, \tag{2.27}$$

respectively. By virtue of the non-secularity condition for $y_{1,0}^{(3)}$, we then obtain

$$G_1(\xi_1, \tau) = 0. \quad (2.28)$$

Accordingly, it follows from (2.23) that

$$\begin{aligned} y_{1,0}^{(3)} &= \frac{m^2}{2\omega_1\omega_2^2(\lambda_2-\lambda_1)} f_1 \frac{\partial \tilde{\Phi}_1}{\partial \xi_2} + h_1(\xi_1, \tau) \\ &= -\frac{2\beta m^2}{2\omega_1^2\omega_2^2(\lambda_1-\lambda_2)^2} f_1 |f_2|^2 + h_1(\xi_1, \tau), \end{aligned} \quad (2.29)$$

where $h_1(\xi_1, \tau)$ is to be determined in the next step. Integrating (2.26) and (2.27), we find

$$\psi_1^{(4)} = \tilde{\Psi}_1(\xi_2, \tau) + \Psi_1(\xi_1, \tau), \quad (2.30)$$

$$\tilde{\Psi}_1(\xi_2, \tau) \equiv \frac{3\beta(1-\lambda_1\lambda_2)}{\omega_1^2(\lambda_1-\lambda_2)^2} \int_0^{\xi_2} |f_2(\xi, \tau)|^2 d\xi, \quad (2.31)$$

and

$$\Omega_1^{(2)} = \tilde{\Theta}_1(\xi_2, \tau) + \Theta_1(\xi_1, \tau), \quad (2.32)$$

$$\tilde{\Theta}_1(\xi_2, \tau) \equiv \int_0^{\xi_2} H_1(\xi, \tau) d\xi, \quad (2.33)$$

where $\Psi_1(\xi_1, \tau)$ and $\Theta_1(\xi_1, \tau)$ are due to the self-interaction and determined in the next order equations. Incidentally, it is evident that

$\tilde{\Psi}_1(\xi_2, \tau)$ and $\tilde{\Theta}_1(\xi_2, \tau)$ are real.

Similarly, the results for $\ell=0$, $n=1$ can be obtained by replacing the subscripts 1,2 and (1,0) with 2,1 and (0,1), respectively, in the above

expressions.

The first term in the right hand side of (2.29), together with $\tilde{\Psi}_I(\xi_2, \tau)$ and $\tilde{\Theta}_I(\xi_2, \tau)$, describes the effect due to the mutual-interaction. The term $\tilde{\Psi}_I(\xi_2, \tau)$ gives the shift of the position of the wave packet, while $\tilde{\Theta}_I(\xi_2, \tau)$ brings about the variations of wave number and frequency as well as $\tilde{\Phi}_I(\xi_2, \tau)$.

The equation (2.26) can be obtained in the following intuitive way : Let $\Delta\tilde{k}_I$ and $\Delta\tilde{\omega}_I$ be the variations of the wave number and the frequency due to the mutual-interaction, respectively, that is, up to the order ε^2 ,

$$\Delta\tilde{k}_I = \varepsilon \frac{\partial \tilde{\Phi}_I}{\partial x} \simeq \varepsilon^2 \frac{\partial \tilde{\Phi}_I}{\partial \xi_2}, \quad \Delta\tilde{\omega}_I = -\varepsilon \frac{\partial \tilde{\Phi}_I}{\partial t} \simeq \lambda_2 \Delta\tilde{k}_I.$$

Then, the variation of the group velocity is given by

$$\begin{aligned} \frac{d(\omega_I + \Delta\tilde{\omega}_I)}{d(k_I + \Delta\tilde{k}_I)} - \lambda_I &\simeq \left(1 - \frac{d}{dk_I} \Delta\tilde{k}_I\right) \left(\frac{d\omega_I}{dk_I} + \frac{d}{dk_I} \Delta\tilde{\omega}_I\right) - \lambda_I \\ &\simeq -(\lambda_I - \lambda_2) \frac{d}{dk_I} \Delta\tilde{k}_I \simeq -\varepsilon^2 (\lambda_I - \lambda_2) \frac{d}{dk_I} \left(\frac{\partial \tilde{\Phi}_I}{\partial \xi_2}\right). \end{aligned}$$

On the other hand, the variation of the velocity of the wave packet due to the mutual-interaction $\Delta\tilde{\lambda}_I$ is obtained as

$$\begin{aligned} \Delta\tilde{\lambda}_I &\simeq \left[\frac{dx}{dt}\right]_{\xi_I = \text{const}} - \lambda_I \\ &\simeq \left[\frac{d}{dt} \varepsilon \psi_I^{(u)}(\xi_I, \xi_2, \tau)\right]_{\xi_I = \text{const}} - \lambda_I \\ &\simeq \varepsilon \frac{\partial \psi_I^{(u)}}{\partial \xi_2} \left[\frac{d\xi_2}{dt}\right]_{\xi_I = \text{const}} \simeq \varepsilon^2 (\lambda_I - \lambda_2) \frac{\partial \psi_I^{(u)}}{\partial \xi_2} \end{aligned}$$

up to the order ε^2 . By equating $\Delta\tilde{\lambda}_i$ to $d(\omega_i + \Delta\tilde{\omega}_i)/d(k_i + \Delta\tilde{k}_i) - \lambda_i$ and by using (2.20), we obtain (2.26).

In conclusion, the nonlinear system consisting of two plane waves can be described, in the lowest order, by the superposition of the renormalized (self-modulated) waves which are governed by the nonlinear Schrödinger equations and undergo the changes in their frequencies, wave numbers and in their orbits due to the mutual-interaction.

§ 3.3 Interaction of Envelope Solitons

If β is positive, the equations (2.21) permit the solutions

$$f_i(\xi_i, \tau) = A_i \operatorname{sech}\left(\sqrt{\frac{3\beta}{2}} \frac{\omega_i A_i}{m} \xi_i\right) \exp\left(i \frac{3\beta A_i^2}{4\omega_i} \tau\right), \quad (i=1, 2) \quad (3.1)$$

In this case, the envelopes of the modulated waves, f_1 and f_2 are of solitary wave type. Such modulated waves are often called "envelope solitons". The envelope soliton solutions play an important role in time-evolution of solutions of the nonlinear Schrödinger equation, that is, they are the remarkably stable entities and many localized initial data disintegrate into a finite number of envelope solitons and an oscillatory tail with decreasing amplitude. Therefore, it is important to study the interaction of envelope solitons. When the two envelope solitons travel with nearly equal velocities, i.e., when the relative velocity of them is of order ϵ , their interaction has been investigated by solving the nonlinear Schrödinger equation (see chap. 1). However, the interaction of two envelope solitons with the relative velocity of the order of unity cannot be described by the single nonlinear Schrödinger equation. Therefore, the new perturbation approach which has been just developed in § 3.2 is required.

Substituting (3.1) into (2.28) and putting $g_i(\xi_i, \tau) = \hat{g}_i(\xi_i) \exp[3i\beta A_i^2 \tau / (4\omega_i)]$, where $\hat{g}_i(\xi_i)$ is real, we obtain the equations

$$\frac{m^2}{2\omega_i^3} \frac{d^2 \hat{g}_i}{d\xi_i^2} + \left(\frac{9\beta}{2\omega_i} \hat{f}_i^2 - \frac{3\beta}{4\omega_i} A_i^2 \right) \hat{g}_i = \hat{f}_i \frac{\partial \hat{E}_i}{\partial \tau} + \frac{m^2}{\omega_i^3} \frac{d^2 \hat{f}_i}{d\xi_i^2} \frac{\partial \psi_i}{\partial \xi_i} + \frac{m^2}{2\omega_i^3} \frac{d\hat{f}_i}{d\xi_i} \frac{\partial^2 \psi_i}{\partial \xi_i^2}, \quad (3.2a)$$

$$\frac{m^2}{2\omega_i^3} f_i \frac{\partial^2 \Phi_i}{\partial \xi_i^2} + \frac{m^2}{\omega_i^2} \frac{d\hat{f}_i}{d\xi_i} \frac{\partial \Phi_i}{\partial \xi_i} + \left(\frac{3\beta\lambda_i A_i^2}{4\omega_i^2} - \frac{\partial \psi_i}{\partial \tau} \right) \frac{d\hat{f}_i}{d\xi_i} = 0, \quad (3.2b)$$

where

$$\hat{f}_i = A_i \operatorname{sech} \left(\sqrt{\frac{3\beta}{2}} \frac{\omega_i A_i}{m} \xi_i \right).$$

Taking the advantage that there exist three unknowns, g_i , Φ_i and ψ_i , in spite of two equations, we can assume both Φ_i and ψ_i to depend on τ only. The equation (3.2b) is then solved to give

$$\psi_i = \frac{3\beta\lambda_i A_i^2}{4\omega_i^2} \tau.$$

Substitution of this into (3.2a) yields

$$\frac{d^2 \hat{g}_i}{d\bar{z}_i^2} + (6 \operatorname{sech}^2 \bar{z}_i - 1) \hat{g}_i = \frac{4\omega_i}{3\beta A_i^2} \frac{d\Phi_i}{d\tau} \hat{f}_i, \quad (3.3)$$

where

$$\bar{z}_i = \sqrt{\frac{3\beta}{2}} \frac{\omega_i A_i}{m} \xi_i.$$

In (3.3), $d\Phi_i/d\tau$ must be constant because \hat{f}_i and \hat{g}_i are independent of τ . If we put

$$\frac{d\Phi_i}{d\tau} = \frac{3\beta A_i^2}{2\omega_i} \alpha_i, \quad (\alpha_i : \text{const}) \quad (3.4)$$

the solution of (3.3) vanishing at $z_i = \pm\infty$ can be easily obtained :

$$\hat{g}_i = A_i \alpha_i \operatorname{sech} z_i - A_i \alpha_i z_i \tanh z_i \operatorname{sech} z_i + C_i \tanh z_i \operatorname{sech} z_i ,$$

where C_i is an arbitrary constant. Without loss of generality, we can take $C_i = 0$ because the arbitrariness of C_i is absorbed in that of the phase constant of the wave packet δ_i . From (3.4) we get

$$\Phi_i = \frac{3\beta A_i^2}{2\omega_i} \alpha_i \tau ,$$

where the unimportant integration constant has been taken to be zero.

We now define a new soliton solution f_i' by putting $A_i(1 + \varepsilon \alpha_i)$ in place of A_i in (3.1), i.e.,

$$f_i' = A_i(1 + \varepsilon \alpha_i) \operatorname{sech} \left\{ \sqrt{\frac{3\beta}{2}} \frac{\omega_i A_i(1 + \varepsilon \alpha_i)}{m} z_i \right\} \exp \left\{ i \frac{3\beta}{4\omega_i} A_i^2 (1 + \varepsilon \alpha_i)^2 \tau \right\} .$$

Expanding separately the amplitude and the phase in powers of ε , we can show that

$$f_i' = \{ f_i + \varepsilon g_i + O(\varepsilon^2) \} \exp \{ i \varepsilon \Phi_i + O(\varepsilon^2) \} .$$

This implies that the arbitrariness of α_i can be absorbed in the amplitude $A_i' = A_i(1 + \varepsilon \alpha_i)$, so that we can take $\alpha_i = 0$ without loss of generality.

Finally, we obtain

$$g_i = 0, \quad \Phi_i = 0, \quad \psi_i = \frac{3\beta \lambda_i A_i^2}{4\omega_i^2} \tau .$$

These results for ψ_1 and ψ_2 can be easily understood : In the solitary wave solutions of (2.21), the frequency shift due to the nonlinear

interaction is given by $\Delta\omega_i = -\varepsilon^2 3\beta A_i^2 / 4\omega_i$ (see (3.1)). The corresponding change in the group velocity is $\Delta\lambda_i = d(\omega_i + \Delta\omega_i) / dk_i - \lambda_i = \varepsilon^2 3\beta A_i^2 \lambda_i / 4\omega_i^2$. In view of these expressions, the modification in the orbit of the envelope soliton is expressed as

It follows from (2.20) and (3.1) that

$$\tilde{\Phi}_1(\xi_2, \tau) = \frac{\sqrt{6\beta} m A_2}{\omega_1 \omega_2 (\lambda_1 - \lambda_2)} \tanh\left(\sqrt{\frac{3\beta}{2}} \frac{\omega_2 A_2}{m} \xi_2\right).$$

Then, the shifts of wave number and frequency due to the mutual-interaction are, up to the order ε^2 , given by

$$\Delta\tilde{k}_1 \equiv \varepsilon \frac{\partial \tilde{\Phi}_1}{\partial x} \simeq \varepsilon^2 \frac{3\beta A_2^2}{\omega_1 (\lambda_1 - \lambda_2)} \operatorname{sech}^2\left(\sqrt{\frac{3\beta}{2}} \frac{\omega_2 A_2}{m} \xi_2\right),$$

$$\Delta\tilde{\omega}_1 \equiv -\varepsilon \frac{\partial \tilde{\Phi}_1}{\partial t} \simeq \varepsilon^2 \frac{3\beta A_2^2 \lambda_2}{\omega_1 (\lambda_1 - \lambda_2)} \operatorname{sech}^2\left(\sqrt{\frac{3\beta}{2}} \frac{\omega_2 A_2}{m} \xi_2\right).$$

It is easily seen that the frequency shift observed in the reference frame travelling with the envelope soliton ($\xi_1 = \text{const}$) is, up to the order ε^2 , independent of λ_1 and λ_2 , i.e.,

$$\Delta\tilde{\omega}_1 - \Delta\tilde{k}_1 \cdot \lambda_1 = -\varepsilon^2 \frac{3\beta A_2^2}{\omega_1} \operatorname{sech}^2\left(\sqrt{\frac{3\beta}{2}} \frac{\omega_2 A_2}{m} \xi_2\right).$$

In the same way, $\tilde{\Phi}_2(\xi_1, \tau)$, $\Delta\tilde{k}_2$ and $\Delta\tilde{\omega}_2$ can be obtained.

Substituting (3.1) into (2.31), we have

$$\tilde{\Psi}_1(\xi_2, \tau) = \frac{\sqrt{6\beta} m A_2 (1 - \lambda_1 \lambda_2)}{\omega_1^2 \omega_2 (\lambda_1 - \lambda_2)^2} \tanh\left(\sqrt{\frac{3\beta}{2}} \frac{\omega_2 A_2}{m} \xi_2\right).$$

The variation of the velocity of the envelope soliton due to the mutual-interaction is given by

$$\Delta \tilde{\lambda}_1 \simeq \varepsilon^2 (\lambda_1 - \lambda_2) \frac{\partial \tilde{\Psi}_1}{\partial \tilde{\xi}_2} = \varepsilon^2 \frac{3\beta A_2^2 (1 - \lambda_1 \lambda_2)}{\omega_1^2 (\lambda_1 - \lambda_2)} \operatorname{sech}^2 \left(\sqrt{\frac{3\beta}{2}} \frac{\omega_2 A_2}{m} \tilde{\xi}_2 \right),$$

where note that $\tilde{\xi}_1$ is kept constant in calculating $\Delta \tilde{\lambda}_1$. By replacing the subscripts 1 and 2 with 2 and 1, respectively, we obtain the expression for $\Delta \tilde{\lambda}_2$. If $\lambda_1 > \lambda_2$, then we have $\Delta \tilde{\lambda}_1 > 0$ and $\Delta \tilde{\lambda}_2 < 0$. Under the same condition, it is easily shown that the shifts of position which the two solitary waves undergo through the whole process of the mutual-interaction (from $t = -\infty$ to $t = +\infty$), δ_1 and δ_2 , are given by

$$\delta_1 \simeq \varepsilon \{ \tilde{\Psi}_1(\tilde{\xi}_2 = \infty) - \tilde{\Psi}_1(\tilde{\xi}_2 = -\infty) \} = \varepsilon \frac{2\sqrt{6\beta} m A_2 (1 - \lambda_1 \lambda_2)}{\omega_1^2 \omega_2 (\lambda_1 - \lambda_2)^2} > 0,$$

$$\delta_2 \simeq \varepsilon \{ \tilde{\Psi}_2(\tilde{\xi}_1 = -\infty) - \tilde{\Psi}_2(\tilde{\xi}_1 = \infty) \} = -\varepsilon \frac{2\sqrt{6\beta} m A_1 (1 - \lambda_1 \lambda_2)}{\omega_1 \omega_2^2 (\lambda_1 - \lambda_2)^2} < 0.$$

Substitution of (3.1) into (2.29) shows that the wave amplitude for the mode $\ell = 1$, $n=0$ ($\ell = 0$, $n=1$) decreases while it is interacting with another mode $\ell = 0$, $n=1$ ($\ell = 1$, $n=0$).

Now, we can summarize the above results as follows : The two envelope solitons pull against each other as they approach, change their amplitudes, frequencies and velocities due to the mutual-interaction and go away regaining their original forms and velocities. We note here that this result is similar to that in the interaction between two envelope solitons with a small relative velocity ($\sim O(\varepsilon)$).

§ 3.4 Generalization to a Wide Class of Nonlinear Systems

We now proceed to apply the perturbation method developed in § 3.2 to a wide class of strongly dispersive and weakly nonlinear systems. Here we consider the equation (I.1.13) which was used by Taniuti and Yajima [41] in the study of the modulation of a plane wave due to the nonlinear self-interaction.

Let $U^{(0)}$ be a constant solution satisfying the relation

$$B(U^{(0)}) = 0. \quad (4.1)$$

Following the discussion in § 3.2, we expand U as

$$U = U^{(0)} + \sum_{\alpha=1}^{\infty} \varepsilon^{\alpha} \sum_{\ell, n=-\infty}^{\infty} U_{\ell, n}^{(\alpha)}(\xi_1, \xi_2, \tau) Z_{\ell, n}, \quad (4.2)$$

where ξ_1 , ξ_2 and τ are the stretched variables introduced by (2.4) and $Z_{\ell, n}$ is given by (2.6b). In order to ensure reality of U , we assume that

$$U_{\ell, n}^{(\alpha)*} = U_{-\ell, -n}^{(\alpha)}, \quad (4.3a)$$

$$\Omega_i^{(j)*} = \Omega_i^{(j)}, \quad \psi_i^{(j)*} = \psi_i^{(j)}, \quad (i=1, 2) \quad (4.3b)$$

Introduction of these into (I.1.13) yields the sequence of equations to be solved, corresponding to the successive powers of ε of the same harmonics.

In the lowest order, we have

$$W_{\ell, n} U_{\ell, n}^{(0)} = 0, \quad (4.4)$$

$$W_{\ell, n} = -i(\ell\omega_1 + n\omega_2)I + i(\ell k_1 + n k_2)A_0 + \nabla B_0, \quad (4.5)$$

where I is the unit matrix, $A_0 = A(U^{(0)})$ and $(\nabla B_0)_{ij} = (\partial B_i / \partial u_j)_{U=U^{(0)}}$.

We now suppose that

$$\left. \begin{aligned} \det W_{\ell,n} &= 0, \text{ for } |\ell| + |n| = 1, \\ &\neq 0, \text{ otherwise.} \end{aligned} \right\} \quad (4.6)$$

The equation (4.4) then yields

$$U_{1,0}^{(1)} = \mathcal{P}_1(\xi_1, \xi_2, \tau) R_1, \quad (4.7a)$$

$$U_{0,1}^{(1)} = \mathcal{P}_2(\xi_1, \xi_2, \tau) R_2, \quad (4.7b)$$

and for $|\ell| + |n| \neq 1$,

$$U_{\ell,n}^{(1)} = 0, \quad (4.8)$$

where R_1 and R_2 are the right eigenvectors of $W_{1,0}$ and $W_{0,1}$, respectively, i.e.,

$$W_{1,0} R_1 = 0 \quad \text{and} \quad W_{0,1} R_2 = 0, \quad (4.9a,b)$$

and \mathcal{P}_1 and \mathcal{P}_2 are scalar functions of ξ_1 , ξ_2 and τ to be determined later.

If we differentiate (4.9a) with respect to k_1 and make use of the definition of λ_1 , i.e., the equation (2.5), we get

$$(\lambda_1 I - A_0) R_1 = -i W_{1,0} \frac{dR_1}{dk_1}. \quad (4.10)$$

Introducing a row vector L_1 corresponding to R_1 through the equation

$$L_1 W_{1,0} = 0, \quad (4.11)$$

and multiplying (4.10) by L_1 from the left, then we have the relation

$$L_1(\lambda_1 I - A_0)R_1 = 0. \quad (4.12a)$$

Again, we differentiate (4.10) with respect to k_1 and make use of the same procedure as the above to obtain

$$L_1(\lambda_1 I - A_0) \frac{dR_1}{dk_1} = -\frac{1}{2} \frac{d^2 \omega_1}{dk_1^2} (L_1 R_1). \quad (4.13a)$$

Similarly, we have

$$L_2(\lambda_2 I - A_0)R_2 = 0, \quad (4.12b)$$

$$L_2(\lambda_2 I - A_0) \frac{dR_2}{dk_2} = -\frac{1}{2} \frac{d^2 \omega_2}{dk_2^2} (L_2 R_2), \quad (4.13b)$$

where L_2 is the left eigenvector, i.e., $L_2^W \omega_{0,1} = 0$. These relations will be used later.

In the next order, we have

$$\begin{aligned} W_{\ell,n} U_{\ell,n}^{(2)} - (\lambda_1 I - A_0) \frac{\partial U_{\ell,n}^{(1)}}{\partial \xi_1} - (\lambda_2 I - A_0) \frac{\partial U_{\ell,n}^{(1)}}{\partial \xi_2} \\ + i \sum_{\ell',n'} (\ell' k_1 + n' k_2) (\nabla A_0 \cdot U_{\ell-\ell',n-n'}^{(1)} U_{\ell',n'}^{(1)} + \frac{1}{2} \sum_{\ell',n'} \nabla \nabla B_0 : U_{\ell-\ell',n-n'}^{(1)} U_{\ell',n'}^{(1)} = 0. \end{aligned} \quad (4.14)$$

It is noted that in the present section the following notations are used :

$$\nabla A_0 \cdot U^{(1)} \equiv \sum_{i=1}^n \left(\frac{\partial A}{\partial u_i} \right)_{U=U^{(1)}} u_i^{(1)}, \quad \nabla \nabla A_0 : U^{(1)} U^{(1)} \equiv \sum_{i,j} \left(\frac{\partial^2 A}{\partial u_i \partial u_j} \right)_{U=U^{(1)}} u_i^{(1)} u_j^{(1)},$$

$$\nabla \nabla B_0 : U^{(1)} U^{(1)} \equiv \sum_{i,j} \left(\frac{\partial^2 B}{\partial u_i \partial u_j} \right)_{U=U^{(1)}} u_i^{(1)} u_j^{(1)},$$

and so on.

In view of (4.7a) and (4.8), (4.14) becomes, for the case $\ell=1$, $n=0$,

$$W_{1,0} U_{1,0}^{(2)} - (\lambda_1 I - A_0) R_1 \frac{\partial \varphi_1}{\partial \xi_1} - (\lambda_2 I - A_0) R_1 \frac{\partial \varphi_1}{\partial \xi_2} = 0. \quad (4.15)$$

If we multiply this by L_1 from the left and use (4.11) and (4.12a), then we get

$$(\lambda_1 - \lambda_2) \frac{\partial \varphi_1}{\partial \xi_2} = 0, \text{ i.e., } \varphi_1 = \varphi_1(\xi_1, \tau). \quad (4.16)$$

Taking (4.10) and (4.16) into account, we can solve (4.15) to obtain

$$U_{1,0}^{(2)} = \rho_1(\xi_1, \xi_2, \tau) R_1 - i \frac{\partial \varphi_1}{\partial \xi_1} \frac{dR_1}{dk_1}. \quad (4.17)$$

The equation (4.16) has, in general, the non-zero solution only for

$$|\ell| + |n| \leq 2 \quad ;$$

$$\left. \begin{aligned} U_{2,0}^{(2)} &= S_1 \varphi_1^2, \quad U_{0,2}^{(2)} = S_2 \varphi_2^2, \quad U_{1,1}^{(2)} = T \varphi_1 \varphi_2, \\ U_{1,-1}^{(2)} &= V \varphi_1 \varphi_2^*, \quad U_{0,0}^{(2)} = X_1 |\varphi_1|^2 + X_2 |\varphi_2|^2, \end{aligned} \right\} \quad (4.18)$$

where

$$S_1 = -W_{2,0}^{-1} \{ i k_1 (\nabla A_0 \cdot R_1) R_1 + \frac{1}{2} \nabla \nabla B_0 : R_1 R_1 \}, \quad (4.19a)$$

$$S_2 = -W_{0,2}^{-1} \{ i k_2 (\nabla A_0 \cdot R_2) R_2 + \frac{1}{2} \nabla \nabla B_0 : R_2 R_2 \}, \quad (4.19b)$$

$$T = -W_{1,1}^{-1} \{ i k_1 (\nabla A_0 \cdot R_2) R_1 + i k_2 (\nabla A_0 \cdot R_1) R_2 + \nabla \nabla B_0 : R_1 R_2 \}, \quad (4.19c)$$

$$V = -W_{1,-1}^{-1} \{ i k_1 (\nabla A_0 \cdot R_2^*) R_1 - i k_2 (\nabla A_0 \cdot R_1) R_2^* + \nabla \nabla B_0 : R_1 R_2^* \}, \quad (4.19d)$$

$$X_j = -W_{0,0}^{-1} \{ i k_j (\nabla A_0 \cdot R_j^*) R_j - i k_j (\nabla A_0 \cdot R_j) R_j^* + \nabla \nabla B_0 : R_j R_j^* \}. \quad (4.19e)$$

In the third order, we have

$$\begin{aligned}
& W_{\ell,n} \Pi_{\ell,n}^{(3)} - (\lambda_1 I - A_0) \frac{\partial \Pi_{\ell,n}^{(2)}}{\partial \xi_1} - (\lambda_2 I - A_0) \frac{\partial \Pi_{\ell,n}^{(2)}}{\partial \xi_2} \\
& + \frac{\partial \Pi_{\ell,n}^{(1)}}{\partial \tau} + [(\lambda_1 I - A_0) \frac{\partial \psi_1^{(0)}}{\partial \xi_1} + (\lambda_2 I - A_0) \frac{\partial \psi_1^{(0)}}{\partial \xi_2}] \frac{\partial \Pi_{\ell,n}^{(1)}}{\partial \xi_1} \\
& + [(\lambda_1 I - A_0) \frac{\partial \psi_2^{(0)}}{\partial \xi_1} + (\lambda_2 I - A_0) \frac{\partial \psi_2^{(0)}}{\partial \xi_2}] \frac{\partial \Pi_{\ell,n}^{(1)}}{\partial \xi_2} \\
& - i\ell [(\lambda_1 I - A_0) \frac{\partial \Omega_1^{(1)}}{\partial \xi_1} + (\lambda_2 I - A_0) \frac{\partial \Omega_1^{(1)}}{\partial \xi_2}] \Pi_{\ell,n}^{(1)} \\
& - i\ell [(\lambda_1 I - A_0) \frac{\partial \Omega_2^{(1)}}{\partial \xi_1} + (\lambda_2 I - A_0) \frac{\partial \Omega_2^{(1)}}{\partial \xi_2}] \Pi_{\ell,n}^{(1)} \\
& + \sum_{\ell',n'} (\nabla A_0 \cdot \Pi_{\ell',n'}^{(1)}) \left(\frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2} \right) \Pi_{\ell-\ell',n-n'}^{(1)} \\
& + i \sum_{\ell',n'} (\ell' k_1 + n' k_2) [(\nabla A_0 \cdot \Pi_{\ell-\ell',n-n'}^{(2)}) \Pi_{\ell',n'}^{(1)} + (\nabla A_0 \cdot \Pi_{\ell',n-n'}^{(1)}) \Pi_{\ell',n'}^{(2)}] \\
& + \frac{i}{2} \sum_{\ell',n'} \sum_{\ell'',n''} (\ell'' k_1 + n'' k_2) (\nabla \nabla A_0 : \Pi_{\ell-\ell'-\ell'',n-n'-n''}^{(1)} \Pi_{\ell',n'}^{(1)} \Pi_{\ell'',n''}^{(1)}) \\
& + \sum_{\ell',n'} \nabla \nabla B_0 : \Pi_{\ell-\ell',n-n'}^{(2)} \Pi_{\ell',n'}^{(1)} + \frac{1}{6} \sum_{\ell',n'} \sum_{\ell'',n''} \nabla \nabla \nabla B_0 : \Pi_{\ell-\ell'-\ell'',n-n'-n''}^{(1)} \Pi_{\ell',n'}^{(1)} \Pi_{\ell'',n''}^{(1)} = 0. \quad (4.20)
\end{aligned}$$

We are now ready to determine \mathcal{F}_i , $\Omega_i^{(0)}$, $\psi_i^{(0)}$ and ρ_i . Multiply (4.20) with $\ell=1$ and $n=0$ by L_1 from the left and make use of (4.7), (4.8), (4.11), (4.12a), (4.13a) and (4.17)~(4.19). Then, we have

$$\begin{aligned}
& (\lambda_1 - \lambda_2) \frac{\partial \rho_1}{\partial \xi_2} - \text{Im}(\beta_1) |\varphi_2|^2 \varphi_1 + \left\{ \frac{\partial \varphi_1}{\partial \tau} - \frac{i}{2} \frac{d^2 \omega_1}{dk_1^2} \frac{\partial^2 \varphi_1}{\partial \xi_1^2} + i \alpha_1 |\varphi_1|^2 \varphi_1 \right\} \\
& + i \left\{ (\lambda_1 - \lambda_2) \frac{\partial \Omega_1^{(u)}}{\partial \xi_2} + \text{Re}(\beta_1) |\varphi_2|^2 \varphi_1 + (\lambda_2 - \lambda_1) \frac{\partial \psi_1^{(u)}}{\partial \xi_2} \frac{\partial \varphi_1}{\partial \xi_1} \right\} = 0, \quad (4.21)
\end{aligned}$$

where $\text{Re}(\beta_1)$ and $\text{Im}(\beta_1)$ are the real and the imaginary part, respectively, and α_1 , β_1 are given by

$$\begin{aligned}
\alpha_1 = & (L_1 R_1)^{-1} L_1 \left\{ k_1 \{ (\nabla A_0 \cdot X_1) R_1 - (\nabla A_0 \cdot S_1) R_1^* + 2(\nabla A_0 \cdot R_1^*) S_1 \right. \\
& + (\nabla \nabla A_0 : R_1 R_1^*) R_1 - \frac{1}{2} (\nabla \nabla A_0 : R_1 R_1) R_1^* \} \\
& \left. - i \{ \nabla \nabla B_0 : (S_1 R_1^* + X_1 R_1) + \frac{1}{2} \nabla \nabla \nabla B_0 : R_1 R_1 R_1^* \} \right\}, \quad (4.22a)
\end{aligned}$$

$$\begin{aligned}
\beta_1 = & (L_1 R_1)^{-1} L_1 \left\{ k_1 \{ (\nabla A_0 \cdot X_2) R_1 + (\nabla A_0 \cdot R_2^*) T + (\nabla A_0 \cdot R_2) V + (\nabla \nabla A_0 : R_2^* R_2) R_1 \right. \\
& + k_2 \{ (\nabla A_0 \cdot V) R_2 - (\nabla A_0 \cdot R_2) V + (\nabla A_0 \cdot R_2^*) T \\
& \left. - (\nabla A_0 \cdot T) R_2^* + (\nabla \nabla A_0 : R_1 R_2^*) R_2 - (\nabla \nabla A_0 : R_1 R_2) R_2^* \} \right. \\
& \left. - i \{ \nabla \nabla B_0 : (X_2 R_1 + T R_2^* + V R_2) + \nabla \nabla \nabla B_0 : R_2 R_2^* R_1 \} \right\}. \quad (4.22b)
\end{aligned}$$

We now suppose that

$$(\lambda_1 - \lambda_2) \frac{\partial \Omega_1^{(u)}}{\partial \xi_2} + \text{Re}(\beta_1) |\varphi_2|^2 = 0, \quad (4.23)$$

$$\frac{\partial \psi_1^{(u)}}{\partial \xi_2} = 0. \quad (4.24)$$

Then, by imposing the non-secularity condition for ρ_1 , we obtain the equation for φ_1 from (4.21) :

$$i \frac{\partial \varphi_1}{\partial \tau} + \frac{1}{2} \frac{d^2 \omega_1}{dk_1^2} \frac{\partial^2 \varphi_1}{\partial \xi_1^2} - \alpha_1 |\varphi_1|^2 \varphi_1 = 0. \quad (4.25)$$

The equation (4.21) then reduces to

$$(\lambda_1 - \lambda_2) \frac{\partial \rho_1}{\partial \xi_1} = \text{Im}(\beta_1) |\varphi_2(\xi_2, \tau)|^2 \varphi_1(\xi_1, \tau). \quad (4.26)$$

It follows from (4.23), (4.24) and (4.26) that

$$\Omega_1^{(1)} = \frac{1}{\lambda_2 - \lambda_1} \text{Re}(\beta_1) \int_{\xi_2}^{\xi_2} |\varphi_2(\xi, \tau)|^2 d\xi + \Phi_1(\xi_1, \tau), \quad (4.27)$$

$$\psi_1^{(0)} = \psi_1(\xi_1, \tau), \quad (4.28)$$

$$\rho_1 = \frac{1}{\lambda_1 - \lambda_2} \text{Im}(\beta_1) \int_{\xi_2}^{\xi_2} |\varphi_2(\xi, \tau)|^2 d\xi \varphi_1(\xi_1, \tau) + g_1(\xi_1, \tau). \quad (4.29)$$

The arbitrary functions $\Phi_1(\xi_1, \tau)$, $\psi_1(\xi_1, \tau)$ and $g_1(\xi_1, \tau)$ are to be determined in the next order.

If α_i is real and $\alpha_i(d^2\omega_i/dk_i^2) < 0$, (4.25) permits the envelope soliton solutions vanishing at $x = \pm\infty$. For a system consisting of two envelope solitons, $\Phi_1(\xi_1, \tau)$ and $g_1(\xi_1, \tau)$ can be absorbed into $\varphi_1(\xi_1, \tau)$ so that these may be taken to be zero. On the other hand, $\psi_1(\xi_1, \tau)$ can be obtained by estimating the change of the group velocity corresponding to the frequency shift due to the nonlinear self-interaction. Such a circumstance has been already described in § 3.3.

The quantity $\psi_1^{(1)}$, which describes the orbit modification of the wave packet, is determined in the fourth order equation. However, this can be found without actually performing complicated higher order calculations. Following the intuitive discussion in the end of § 3.2, we then get

$$\psi_1^{(1)} = \frac{d}{dk_1} \left(\frac{\text{Re}(\beta_1)}{\lambda_1 - \lambda_2} \right) \int_{\xi_2}^{\xi_2} |\varphi_2(\xi, \tau)|^2 d\xi. \quad (4.30)$$

The results for $\ell = 0$ and $n=1$ are obtained by replacing the subscript 1 with 2, and vice versa, in (4.22) ~ (4.30).

If α_i and β_i are real, we can say that in the asymptotic sense, a nonlinear and strongly dispersive system consisting of two plane waves is described by the superposition of the self-modulated waves in the lowest order approximation if the shifts in position of the modulated waves and those in the wave phases due to the mutual interaction are taken into account.

Recently, Yuen and Lake have made the experiments on the interaction of two envelope solitons in deep water waves and observed that the two envelope solitons with the different carrier frequencies pass through each other.

Chapter 4

A Class of Exactly Solvable Nonlinear Evolution

Equations

§ 4.1 Introduction

In 1967 GGKM [34] discovered the inverse scattering method to give the scheme for solving the initial value problem for the KdV equation. Lax [29] generalized their ideas to show that the method is applicable to nonlinear evolution equations other than the KdV equation. In fact, following Lax's approach, Zakharov and Shabat [59] solved exactly the nonlinear Schrödinger equation. Since then, the method has been applied to many other nonlinear evolution equations. These circumstances have been already viewed in chap. 1. It enabled us not only to solve the initial value problems of some class of nonlinear evolution equations but to reveal the characteristic features of nonlinear (dispersive) waves.

In applying the method, we must first find the linear eigenvalue problem the eigenvalues of which remain invariant as the potential evolves according to the given nonlinear evolution equation. This still remains a highly nontrivial step. However, it is relatively easy to find the evolution equations which can be solved by means of a given linear eigenvalue problem. AKNS [109],[110] found a broad class of nonlinear evolution equations which can be solved by means of the generalized Zakharov-Shabat eigenvalue problem (I.2.37). On the other hand, Zakharov and Manakov [116] showed that the system of equations describing the three-wave resonant interaction can be reduced to a third order eigenvalue

problem. Here following the method presented in [109], we investigate what class of nonlinear evolution equations can be solved by means of the third order eigenvalue problem. In the next section, we present some interesting examples of them. One of the examples can be considered to be a model of the interaction between Langmuir waves and ion-sound waves in a plasma.

§ 4.2 Method and Results

AKNS's method has been already outlined in chap. 1. We note that their eigenvalue equation (I.2.37) and the time evolution of the eigenfunctions (I.2.38) can be rewritten by use of the Pauli spin matrices σ_i ($i=1,2,3$) as follows :

$$\frac{\partial V}{\partial x} = (i\zeta\sigma_3 + \sum_{i=1}^2 a_i(x, t)\sigma_i) V, \quad (2.1)$$

$$\frac{\partial V}{\partial t} = (\sum_{i=1}^3 b_i(x, t, \zeta) \sigma_i) V, \quad (2.2)$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here we extend (2.1) to the third order eigenvalue problem to investigate what class of nonlinear evolution equations can be solved by means of it. For this purpose, it is convenient to introduce a matrix representation of the generators of $SU(3)$, λ_i ($i=1,2, \dots, 8$), in place of σ_i . The λ_i satisfy the commutation relations

$$[\lambda_i, \lambda_j] = 2i \sum_{k=1}^8 f_{ijk} \lambda_k, \quad (i, j = 1, 2, \dots, 8) \quad (2.3)$$

where f_{ijk} is fully antisymmetric. The following representation is taken here :

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Then, the nonvanishing values of f_{ijk} are permutations of the following :

$$\left. \begin{aligned} f_{123} &= 1, \quad f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, \\ f_{458} &= f_{678} = \frac{1}{2}\sqrt{3}. \end{aligned} \right\} \quad (2.4)$$

Now we take the eigenvalue problem and the time evolution of the eigenfunctions as follows :

$$\frac{\partial V}{\partial x} = (\zeta(\lambda_3 + \mu\lambda_8) + \sum_{\substack{i=1 \\ (i \neq 3)}}^7 a_i(x, t) \lambda_i) V, \quad (2.5)$$

$$\frac{\partial V}{\partial t} = (\sum_{i=1}^8 \theta_i(x, t, \zeta) \lambda_i) V, \quad (2.6)$$

where μ is a constant.

By cross differentiating (2.5) and (2.6) and by making use of (2.3), we obtain the conditions for the time invariance of ζ ,

$$\frac{\partial a_i}{\partial t} - \frac{\partial \theta_i}{\partial x} + 2i\zeta \sum_{j=1}^8 (f_{3ji} + \mu f_{8ji}) \theta_j + 2i \sum_{\substack{k,j=1 \\ (j \neq 3,8)}}^8 f_{ijk} a_j \theta_k = 0. \quad (2.7)$$

By substituting the finite expansion of b_i , $\theta_i = \sum_{n=0}^N \zeta^n \theta_i^{(n)}$, into (2.7) and by equating the coefficients of various powers of ζ to zero,

we obtain the sequence of the equations for $b_i^{(n)}$. We can solve these equations successively by starting from $b_i^{(N)}=0$ ($i=1,2,4, \dots, 7$) and finally obtain the evolution equations for $a_i(x,t)$ ($i=1,2,4 \sim 7$) as the compatibility condition. If we put $a_i=0$ ($i=4, \dots, 7$), $b_i=0$ ($i=4, \dots, 8$) and $\mu=0$, it is easily seen that the results of AKNS are obtained.

We present only some interesting examples of the nonlinear evolution equations thus obtained because the general results are lengthy. In what follows, u_1, u_2, \dots, w_2 are defined by the equations

$$\left. \begin{aligned} u_1 &= a_1 + i a_2, & v_1 &= a_4 + i a_5, & w_1 &= a_6 + i a_7, \\ u_2 &= a_1 - i a_2, & v_2 &= a_4 - i a_5, & w_2 &= a_6 - i a_7. \end{aligned} \right\} \quad (2.8)$$

(i) $N=1$. Let us take

$$\left. \begin{aligned} \theta_1 &= -c_1 a_1, & \theta_2 &= -c_1 a_2, & \theta_3 &= \zeta A, & \theta_4 &= -c_2 a_4, \\ \theta_5 &= -c_2 a_5, & \theta_6 &= -c_3 a_6, & \theta_7 &= -c_3 a_7, & \theta_8 &= \zeta B, \end{aligned} \right\} \quad (2.9)$$

where $c_1 = -A$, $c_2 = -(A + \sqrt{3}B)/(1 + \sqrt{3}\mu)$, $c_3 = -(A - \sqrt{3}B)/(1 - \sqrt{3}\mu)$ and A, B, μ are arbitrary constants. Then, we have

$$\left. \begin{aligned} \frac{\partial u_1}{\partial t} + c_1 \frac{\partial u_1}{\partial x} &= (c_2 - c_3) v_1 w_2, & \frac{\partial u_2}{\partial t} + c_1 \frac{\partial u_2}{\partial x} &= -(c_2 - c_3) v_2 w_1, \\ \frac{\partial v_1}{\partial t} + c_2 \frac{\partial v_1}{\partial x} &= -(c_3 - c_1) u_1 w_1, & \frac{\partial v_2}{\partial t} + c_2 \frac{\partial v_2}{\partial x} &= (c_3 - c_1) u_2 w_2, \\ \frac{\partial w_1}{\partial t} + c_3 \frac{\partial w_1}{\partial x} &= (c_1 - c_2) u_2 v_1, & \frac{\partial w_2}{\partial t} + c_3 \frac{\partial w_2}{\partial x} &= -(c_1 - c_2) u_1 v_2. \end{aligned} \right\} \quad (2.10)$$

Suppose that A, B and μ are real and take as follows :

$$u_1 = \varepsilon_2 \varepsilon_3 u_2^*, \quad v_2 = -\varepsilon_3 \varepsilon_1 v_1^*, \quad w_1 = \varepsilon_1 \varepsilon_2 w_2^*,$$

$$u_2 = i\varepsilon_1 \alpha g_1, \quad v_1 = -i\varepsilon_2 \beta g_2, \quad w_2 = i\varepsilon_3 \gamma g_3,$$

where $\alpha = [(c_3 - c_1)(c_2 - c_1)]^{-1/2}$, $\beta = [(c_3 - c_2)(c_2 - c_1)]^{-1/2}$,

$\gamma = [(c_3 - c_1)(c_3 - c_2)]^{-1/2}$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 = \pm 1$, and it has been assumed that $c_1 < c_2 < c_3$. Then, (2.10) reduces to the following equations and their complex conjugates,

$$\frac{\partial g_1}{\partial t} + c_1 \frac{\partial g_1}{\partial x} = i\varepsilon_1 g_2^* g_3^*, \quad (2.11a)$$

$$\frac{\partial g_2}{\partial t} + c_2 \frac{\partial g_2}{\partial x} = i\varepsilon_2 g_3^* g_1^*, \quad (2.11b)$$

$$\frac{\partial g_3}{\partial t} + c_3 \frac{\partial g_3}{\partial x} = i\varepsilon_3 g_1^* g_2^*. \quad (2.11c)$$

This system of equations describes the three-wave resonant interaction [122] and q_1, q_2, q_3 are the (complex) envelopes and c_1, c_2, c_3 are the respective group velocities. When we choose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \pm 1$, we have the case of the explosive type of interaction. If we choose one of the ε_i 's different from the others, we have the case of the decay type of interaction.

The inverse scattering form for (2.11) was first found by Zakharov and Manakov [116] and recently the inverse scattering problem for the case $q_1, q_2, q_3 \rightarrow 0$ at infinity was solved by Kaup [126]. It is noted that the case that two of the three group velocities are the same can be reduced to the second order eigenvalue problem. We also note that (2.10) is the general result for $N=1$.

(ii) $N=2$. Case a) Let us take $a_6=a_7=0$ and $\sqrt{3}\mu=1$,

$$\theta_1 = -i\zeta a_1 - \frac{1}{2} \frac{\partial a_2}{\partial x}, \quad \theta_2 = -i\zeta a_2 + \frac{1}{2} \frac{\partial a_1}{\partial x},$$

$$\theta_3 = -i\zeta^2 + \frac{i}{2}(a_1^2 + a_2^2) + \frac{i}{4}(a_4^2 + a_5^2),$$

$$\theta_4 = -i\zeta a_4 - \frac{1}{2} \frac{\partial a_5}{\partial x}, \quad \theta_5 = -i\zeta a_5 + \frac{1}{2} \frac{\partial a_4}{\partial x}, \quad \theta_6 = -\frac{1}{2}(a_1 a_4 + a_2 a_5),$$

$$\theta_7 = -\frac{i}{2}(a_1 a_5 - a_2 a_4), \quad \sqrt{3}\theta_8 = -i\zeta^2 + \frac{3}{4}i(a_4^2 + a_5^2).$$

Then, we have

$$i \frac{\partial u_1}{\partial t} + \frac{1}{2} \frac{\partial^2 u_1}{\partial x^2} - (u_1 u_2 + v_1 v_2) u_1 = 0, \quad (2.12a)$$

$$i \frac{\partial u_2}{\partial t} - \frac{1}{2} \frac{\partial^2 u_2}{\partial x^2} + (u_1 u_2 + v_1 v_2) u_2 = 0, \quad (2.12b)$$

$$i \frac{\partial v_1}{\partial t} + \frac{1}{2} \frac{\partial^2 v_1}{\partial x^2} - (v_1 v_2 + u_1 u_2) v_1 = 0, \quad (2.12c)$$

$$i \frac{\partial v_2}{\partial t} - \frac{1}{2} \frac{\partial^2 v_2}{\partial x^2} + (v_1 v_2 + u_1 u_2) v_2 = 0. \quad (2.12d)$$

When $u_1=u_2^*=u$, $v_1=v_2^*=v$, (2.12) reduces to

$$i \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - (u u^* + v v^*) u = 0, \quad (2.13a)$$

$$i \frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} - (u u^* + v v^*) v = 0. \quad (2.13b)$$

If $u_1=-u_2^*=u$, $v_1=-v_2^*=v$, we have

$$i \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + (uu^* + vv^*)u = 0, \quad (2.14a)$$

$$i \frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + (uu^* + vv^*)v = 0. \quad (2.14b)$$

If $u_1 = u_2^* = u$, $v_1 = -v_2^* = v$, (2.12) becomes

$$i \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - (uu^* - vv^*)u = 0, \quad (2.15a)$$

$$i \frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} - (uu^* - vv^*)v = 0. \quad (2.15b)$$

There three systems of equations are the coupled systems of the two kinds of the nonlinear Schrödinger equation. Any of these systems has solitary wave solutions and in particular, the system (2.15) has three kinds of solitary wave solutions according to boundary conditions. The systems (2.13) and (2.14) may describe the nonlinear modulation of the electromagnetic waves with arbitrary polarization. Recently, Manakov [150] solved (2.14) by means of the inverse scattering method to show that the amplitudes and velocities of the solitary waves do not change after the interaction but the polarizations change in general.

Case b) Let us take $\mu = \sqrt{3}$ and

$$\theta_1 = \zeta A_2 a_1 - \frac{i}{2} A_2 \frac{\partial a_2}{\partial x} + A_1 a_1 + \frac{1}{2} A_2 (a_4 a_6 + a_5 a_7),$$

$$\theta_2 = \zeta A_2 a_2 + \frac{i}{2} A_2 \frac{\partial a_1}{\partial x} + A_1 a_2 + \frac{1}{2} A_2 (a_5 a_6 - a_4 a_7),$$

$$\theta_3 = \zeta^2 A_2 + \zeta A_1 - \frac{1}{2} A_2 (a_1^2 + a_2^2) - \frac{1}{4} A_2 (a_6^2 + a_7^2),$$

$$\begin{aligned}
\theta_4 &= A_1 a_4 + \frac{1}{2} A_2 (a_1 a_6 - a_2 a_7), \\
\theta_5 &= A_1 a_5 + \frac{1}{2} A_2 (a_1 a_7 + a_2 a_6), \\
\theta_6 &= -\zeta A_2 a_6 + \frac{i}{2} A_2 \frac{\partial a_7}{\partial x} + A_1 a_6 + \frac{1}{2} A_2 (a_1 a_4 + a_2 a_5), \\
\theta_7 &= -\zeta A_2 a_7 - \frac{i}{2} A_2 \frac{\partial a_6}{\partial x} + A_1 a_7 + \frac{1}{2} A_2 (a_1 a_5 - a_2 a_4), \\
\theta_8 &= -\zeta^2 \frac{A_2}{\sqrt{3}} + \sqrt{3} \zeta A_1 + \frac{\sqrt{3}}{4} A_2 (a_6^2 + a_7^2).
\end{aligned}$$

If we put $u_2 = w_2 = 0$, $v_2 = C$ (:const.) in the results obtained, then we have

$$\frac{\partial u_1}{\partial t} + \frac{A_2}{2} \frac{\partial^2 u_1}{\partial x^2} - A_1 \frac{\partial u_1}{\partial x} - \frac{A_2}{2} C v_1 v_1 = 0, \quad (2.16a)$$

$$\frac{\partial w_1}{\partial t} - \frac{A_2}{2} \frac{\partial^2 w_1}{\partial x^2} - A_1 \frac{\partial w_1}{\partial x} + \frac{A_2}{2} C v_1 w_1 = 0, \quad (2.16b)$$

$$\frac{\partial v_1}{\partial t} - A_1 \frac{\partial v_1}{\partial x} - A_2 \frac{\partial}{\partial x} (u_1 w_1) = 0, \quad (2.16c)$$

where A_1, A_2 are arbitrary constants. Further, if we take $A_1 = -1, A_2 = -i$, $u_1 = w_1^* = \phi$, $v_1 = in$ and $C = -2i$, we obtain

$$i \left(\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \right) + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} - n \phi = 0, \quad (2.17a)$$

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} + \frac{\partial}{\partial x} |\phi|^2 = 0. \quad (2.17b)$$

This system of equations is related to the interactions between Langmuir waves and ion-sound wave in a plasma and will be solved in the next chapter.

(iii) $N=3$. Let us take $a_6=a_7=0$, $\sqrt{3}\mu=1$ and

$$\mathcal{B}_1 = -4\zeta^2 a_1 + 2i\zeta \frac{\partial a_2}{\partial x} - \frac{\partial^2 a_1}{\partial x^2} + 2(a_1^2 + a_2^2)a_1 + 2(a_4^2 + a_5^2)a_1,$$

$$\mathcal{B}_2 = -4\zeta^2 a_2 - 2i\zeta \frac{\partial a_1}{\partial x} - \frac{\partial^2 a_2}{\partial x^2} + 2(a_1^2 + a_2^2)a_2 + 2(a_4^2 + a_5^2)a_2,$$

$$\mathcal{B}_3 = -4\zeta^3 + \zeta\{2(a_1^2 + a_2^2) + (a_4^2 + a_5^2)\} - 2i(a_1 \frac{\partial a_2}{\partial x} - a_2 \frac{\partial a_1}{\partial x}),$$

$$\mathcal{B}_4 = -4\zeta^2 a_4 + 2i\zeta \frac{\partial a_5}{\partial x} - \frac{\partial^2 a_4}{\partial x^2} + 2(a_4^2 + a_5^2)a_4 + 2(a_1^2 + a_2^2)a_4,$$

$$\mathcal{B}_5 = -4\zeta^2 a_5 - 2i\zeta \frac{\partial a_4}{\partial x} - \frac{\partial^2 a_5}{\partial x^2} + 2(a_4^2 + a_5^2)a_5 + 2(a_1^2 + a_2^2)a_5,$$

$$\mathcal{B}_6 = -2\zeta(a_1 a_4 + a_2 a_5) - i(a_5 \frac{\partial a_1}{\partial x} - a_4 \frac{\partial a_2}{\partial x}) + i(a_1 \frac{\partial a_5}{\partial x} - a_2 \frac{\partial a_4}{\partial x}),$$

$$\mathcal{B}_7 = -2\zeta(a_1 a_5 - a_2 a_4) + i(a_4 \frac{\partial a_1}{\partial x} + a_5 \frac{\partial a_2}{\partial x}) - i(a_1 \frac{\partial a_4}{\partial x} + a_2 \frac{\partial a_5}{\partial x}),$$

$$\sqrt{3}\mathcal{B}_8 = -4\zeta^3 + 3\zeta(a_4^2 + a_5^2) - 3i(a_4 \frac{\partial a_5}{\partial x} - a_5 \frac{\partial a_4}{\partial x}).$$

Then we have

$$\frac{\partial u_1}{\partial t} + \frac{\partial^3 u_1}{\partial x^3} - 6u_1 u_2 \frac{\partial u_1}{\partial x} - 3v_2 \frac{\partial}{\partial x}(u_1 v_1) = 0, \quad (2.18a)$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial^3 u_2}{\partial x^3} - 6u_1 u_2 \frac{\partial u_2}{\partial x} - 3v_1 \frac{\partial}{\partial x}(u_2 v_2) = 0, \quad (2.18b)$$

$$\frac{\partial v_1}{\partial t} + \frac{\partial^3 v_1}{\partial x^3} - 6v_1 v_2 \frac{\partial v_1}{\partial x} - 3u_2 \frac{\partial}{\partial x}(u_1 v_1) = 0, \quad (2.18c)$$

$$\frac{\partial v_2}{\partial t} + \frac{\partial^3 v_2}{\partial x^3} - 6v_1 v_2 \frac{\partial v_2}{\partial x} - 3u_1 \frac{\partial}{\partial x}(u_2 v_2) = 0. \quad (2.18d)$$

If we take $u_1 = \mp u_2 = u$, $v_1 = \mp v_2 = v$, then (2.18) reduces to

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} \pm 6u^2 \frac{\partial u}{\partial x} \pm 3v \frac{\partial}{\partial x}(uv) = 0, \quad (2.19a)$$

$$\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} \pm 6v^2 \frac{\partial v}{\partial x} \pm 3u \frac{\partial}{\partial x}(uv) = 0. \quad (2.19b)$$

These may be called the coupled modified KdV equations. For the case $u_1 = -u_2 = u$, $v_1 = -v_2 = v$, (2.19) can be easily solved according to Manakov [150] or Kaup [126] under the boundary condition that u, v tend to zero as $x \rightarrow \pm\infty$. The behaviour of the solutions is analogous to that of (2.14). If we put $u_2 = -1$, $v_2 = -1$ or $u_2 = -1$, $v_2 = 1$, then (2.18) becomes the systems which may be called the coupled KdV equations. However, they are reduced to the KdV equation for $u_1 + v_1$ or $u_1 - v_1$, respectively.

We have examined up to the case $N=5$, but we could not have obtained the interesting examples. It is noted that similar extension was made by Ablowitz and Haberman [192]. They could find that the Boussinesq equation (I.2.41) is included in such classes of nonlinear evolution equations, but they could not find (2.17). Another interesting example which can be solved by the inverse problem of the third order eigenvalue equation was recently found by Newell [193].

Chapter 5

Formation and Interaction of Sonic-Langmuir Solitons

— Inverse Scattering Method —

§ 5.1 Introduction

In Langmuir turbulence the energy of wave field is concentrated in the long-wave part of the spectrum through the nonlinear processes such as nonlinear Landau damping by the electrons and ions, decay of the Langmuir waves with production of ion sound waves and four-plasmon interaction. Since the linear damping mechanisms have little effect on long waves, what mechanism makes the wave energy dissipate to determine the turbulent spectrum comes into question. In this context, Vedenov and Rudakov [194] showed that Langmuir turbulence with sufficiently long wavelength is unstable to spatial modulation. The nonlinear stage of this instability was studied by Zakharov [195], who proposed that the three-dimensional focusing of Langmuir waves called "collapse" is the main energy dissipation mechanism of Langmuir turbulence in the long-wave region.

One-dimensional self-modulation for Langmuir waves does not cause their collapse, but leads to soliton formation. Therefore, one-dimensional Langmuir turbulence may be described by an ensemble of solitons [196]. In this connection, Degtyarev, Makhan'kov and Rudakov [197] proposed a new theory on one-dimensional Langmuir turbulence based on the results of numerical computations.

The purpose of the present chapter is to deal strictly with the interactions of one-dimensional Langmuir waves with sound waves propagating in one-direction, in particular, with the phenomena being concerned

with the sonic-Langmuir solitons by making use of the inverse scattering method.

The system of equations for the ion sound wave under the action of the ponderomotive force due to high-frequency field and for the Langmuir wave was formulated by Zakharov [195] :

$$i \frac{\partial E}{\partial t} + \frac{1}{2} \frac{\partial^2 E}{\partial x^2} - nE = 0, \quad (1.1)$$

$$\frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} - 2 \frac{\partial^2 |E|^2}{\partial x^2} = 0, \quad (1.2)$$

where $E e^{-i\omega_p t}$ is the normalized electric field of the Langmuir oscillation and n the normalized density perturbation. The spatial variable x and the time variable t are also normalized appropriately. This system of equations has the soliton solution,

$$E = \sqrt{(1-v^2)/2} N \operatorname{sech}[N(x-vt)] e^{i v x + i(N^2 v^2)t/2}, \quad (1.3a)$$

$$n = -N^2 \operatorname{sech}^2[N(x-vt)], \quad (1.3b)$$

where N and v are constants. Degtyarev et al. [197] found many interesting phenomena by solving the system of equations (1.1) and (1.2) numerically. These are, for example, the soliton formation from a given initial disturbance ($E \neq 0$, $n=0$), soliton scattering with the emission of ion sound waves, the fusion of two solitons and the fission of a soliton by the absorption of ion sound waves. According to this numerical computation, Langmuir solitons are created or annihilated through the interactions with ion sound. The selection rule with respect to such interactions of Langmuir

solitons with ion sound waves was investigated by Thornhill and ter Haar [198] and by Gibbons, Thornhill, Wardrop and ter Haar [199] with the help of the conservation laws for the system of equations (1.1) and (1.2).

The system can be solved exactly if it is simplified. We consider the ion sound wave propagating in only one-direction, for example, in the positive x-direction, then we may assume that

$$\frac{\partial \eta}{\partial t} \cong -\frac{\partial \eta}{\partial x}, \quad (1.4)$$

where the sound speed is normalized to unity. By using this, we obtain

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \right) \cong -2 \frac{\partial}{\partial x} \left(\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \right).$$

It follows from this that (1.2) can be rewritten as

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} + \frac{\partial |E|^2}{\partial x} = 0. \quad (1.5)$$

The soliton solution of equations (1.1) and (1.5) can be obtained by replacing $\sqrt{(1-v^2)/2}$ in (1.3a) with $\sqrt{1-v}$ on account of $v \approx 1$.

Applying an asymptotic perturbation method to (1.1) and (1.5), Karpman [200] studied the dynamics of soliton formation. The equations (1.1) and (1.5) can be reduced to the system (IV.2.17) through the transformation

$$\phi(x, t) = E(x, t) e^{i(\frac{t}{2} - x)}. \quad (1.6)$$

Therefore, the system of equations (1.1) and (1.5) can be solved exactly by means of the inverse scattering method.

In § 5.2, the eigenvalue equations corresponding to (1.1) and (1.5) are given and the properties of the scattering problem are described. In § 5.3, the Gel'fand-Levitan equations are derived and it is shown how the solutions to (1.1) and (1.5) are represented by the scattering matrix. The zeros of a diagonal element of the scattering matrix correspond to the soliton solutions. The N-soliton solutions are discussed in § 5.4. The phase shifts due to the collisions of solitons are found. The phenomena such as soliton formation and pick-up process of Langmuir wave from a soliton by negative amplitude ion sound are investigated in § 5.5. Section 5.6 is assigned to the derivation of an infinite number of the constants of motion.

§ 5.2 Direct Scattering Problem

The eigenvalue equations and the equations for the time evolution of the eigenfunction associated with (1.1) and (1.5) can be obtained at once from the results given in the previous chapter. However, in order to solve the direct and inverse scattering problem, it is more convenient to use the following transformed equations.

We consider the following eigenvalue equations with $n(x, t)$ and $\phi(x, t)$ as the potentials,

$$\frac{\partial f}{\partial x} + \frac{i}{2\zeta} \begin{pmatrix} n & -2i\zeta\phi^* & n \\ i\phi & 0 & i\phi \\ -n & 2i\zeta\phi^* & -n \end{pmatrix} f = \begin{pmatrix} 3i\zeta & 0 & 0 \\ 0 & i\zeta & 0 \\ 0 & 0 & -i\zeta \end{pmatrix} f, \quad (2.1)$$

where f is a column vector with three components, n is real and E is given by (1.6). The time evolution of the eigenfunction f is given by

$$\frac{\partial f}{\partial t} = \begin{pmatrix} i(\frac{2}{3}\zeta^2 - 2\zeta) & 0 & 0 \\ 0 & -\frac{4}{3}i\zeta^2 & 0 \\ 0 & 0 & i(\frac{2}{3}\zeta^2 + 2\zeta) \end{pmatrix} f + Df, \quad (2.2)$$

$$D = \frac{i}{2\zeta} \begin{pmatrix} n + \frac{|\phi|^2}{2} & -2i\zeta(-\zeta\phi^* + \frac{i}{2}\phi_x^* + \phi^*) & n + \frac{|\phi|^2}{2} \\ -i(\zeta\phi + \frac{i}{2}\phi_x - \phi) & 0 & -i(-\zeta\phi + \frac{i}{2}\phi_x - \phi) \\ -\left(n + \frac{|\phi|^2}{2}\right) & 2i\zeta(\zeta\phi^* + \frac{i}{2}\phi_x^* + \phi^*) & -\left(n + \frac{|\phi|^2}{2}\right) \end{pmatrix}. \quad (2.3)$$

We can make sure directly by cross differentiation of (2.1) and (2.2) that the system of equations (1.1) and (1.5) can be obtained as the condition of time invariance of ζ . The initial value problem for (1.1) and (1.5) is therefore reduced to the direct and inverse scattering problem for (2.1). We assume in what follows that E (so that ϕ) and n vanish sufficiently rapidly as $x \rightarrow \pm\infty$.

The inverse scattering problem of the third order eigenvalue equation was solved for a special case by Kaup [126]. We proceed in parallel with Kaup's work but take care of the singularity ζ^{-1} in (2.1).

We define, for real ζ , the Jost functions $\varphi^{(i)}(x, \zeta)$ and $\psi^{(i)}(x, \zeta)$ ($i=1,2,3$) which are solutions of (2.1) satisfying the boundary conditions

$$\varphi^{(1)}(x, \zeta) e^{-3i\zeta x} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi^{(2)}(x, \zeta) e^{-i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varphi^{(3)}(x, \zeta) e^{i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as $x \rightarrow -\infty$, (2.4)

$$\psi^{(1)}(x, \zeta) e^{-3i\zeta x} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^{(2)}(x, \zeta) e^{-i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi^{(3)}(x, \zeta) e^{i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

as $x \rightarrow +\infty$. (2.5)

We now introduce the Wronskian by

$$W(f, g, h) \equiv \det \begin{pmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{pmatrix}. \quad (2.6)$$

Evidently, $W(f,g,h) \neq 0$ is the necessary and sufficient condition for f, g and h to be linearly independent. When f, g and h are solutions of (2.1), it is shown that

$$\frac{\partial W}{\partial x} = 3i\zeta W. \quad (2.7)$$

From (2.4) ~ (2.7), we get

$$W(\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}) = W(\psi^{(1)}, \psi^{(2)}, \psi^{(3)}) = e^{3i\zeta x} \quad (2.8)$$

Therefore, $\{\varphi^{(j)}(x, \zeta)\}$ and $\{\psi^{(j)}(x, \zeta)\}$ are a set of three linearly independent solutions of (2.1). Consequently, $\varphi^{(j)}(x, \zeta)$ can be represented by a linear combination of $\{\psi^{(k)}(x, \zeta)\}$, that is,

$$\varphi^{(j)}(x, \zeta) = \sum_{k=1}^3 a_{jk}(\zeta) \psi^{(k)}(x, \zeta), \quad (2.9)$$

where $(a_{jk}(\zeta))$ is called the scattering matrix. From (2.8) and (2.9), we obtain

$$\det(a_{jk}(\zeta)) = 1. \quad (2.10)$$

Then, (2.9) can be rewritten as

$$\psi^{(k)}(x, \zeta) = \sum_{j=1}^3 b_{kj}(\zeta) \varphi^{(j)}(x, \zeta), \quad (2.11)$$

where $(b_{kj}(\zeta))$ is the inverse matrix of $(a_{jk}(\zeta))$,

$$\sum_{k=1}^3 a_{jk}(\zeta) b_{kl}(\zeta) = \delta_{jl}. \quad (2.12)$$

By using the boundary conditions (2.4) and (2.5), it follows from (2.9) and (2.11) that

$$\left. \begin{aligned} a_{11}(\zeta) &= \lim_{x \rightarrow \infty} \varphi_1^{(1)} e^{-3i\zeta x}, & a_{33}(\zeta) &= \lim_{x \rightarrow \infty} \varphi_3^{(3)} e^{i\zeta x}, \\ b_{11}(\zeta) &= \lim_{x \rightarrow -\infty} \psi_1^{(1)} e^{-3i\zeta x}, & b_{33}(\zeta) &= \lim_{x \rightarrow -\infty} \psi_3^{(3)} e^{i\zeta x}. \end{aligned} \right\} \quad (2.13)$$

If $n(x)$ and $\phi(x)$ tend to zero faster than $|x|^{-1}$ as $|x| \rightarrow \infty$, we can show the following analytical properties by the Neumann series expansions of the Jost functions $\varphi^{(i)}(x, \zeta)$ and $\psi^{(i)}(x, \zeta)$: $\varphi^{(1)} e^{-3i\zeta x}$, $\psi^{(3)} e^{i\zeta x}$, $a_{11}(\zeta)$ and $b_{33}(\zeta)$ are analytic functions of ζ in the lower half ζ -plane ($\text{Im}(\zeta) < 0$) and $\psi^{(1)} e^{-3i\zeta x}$, $\varphi^{(3)} e^{i\zeta x}$, $a_{33}(\zeta)$ and $b_{11}(\zeta)$ are analytic functions of ζ in the upper half ζ -plane ($\text{Im}(\zeta) > 0$).

The asymptotic properties for large $|\zeta|$ are obtained by the asymptotic expansions of the solutions of (2.1) :

$$\psi^{(1)} e^{-3i\zeta x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + O(\zeta^{-1}) \quad \text{for } \text{Im}(\zeta) > 0, \quad (2.14a)$$

$$\psi^{(3)} e^{i\zeta x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(\zeta^{-1}) \quad \text{for } \text{Im}(\zeta) < 0, \quad (2.14b)$$

$$\varphi^{(1)} e^{-3i\zeta x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + O(\zeta^{-1}) \quad \text{for } \text{Im}(\zeta) < 0, \quad (2.14c)$$

$$\varphi^{(3)} e^{i\zeta x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(\zeta^{-1}) \quad \text{for } \text{Im}(\zeta) > 0. \quad (2.14d)$$

From (2.13) and (2.14), we further obtain

$$a_{33}(\zeta) = 1 + O(\zeta^{-1}), \quad b_{11}(\zeta) = 1 + O(\zeta^{-1}) \quad \text{for } \operatorname{Im}(\zeta) > 0, \quad (2.15a)$$

$$a_{11}(\zeta) = 1 + O(\zeta^{-1}), \quad b_{33}(\zeta) = 1 + O(\zeta^{-1}) \quad \text{for } \operatorname{Im}(\zeta) < 0. \quad (2.15b)$$

Though it is difficult to write explicitly the asymptotic forms of the Jost functions for small $|\zeta|$, it is easily seen that they have merely at most the simple pole at $\zeta=0$ due to the singularity ζ^{-1} in the potential matrix of (2.1).

The analytical properties of the Jost functions $\varphi^{(2)}(x, \zeta)$ and $\psi^{(2)}(x, \zeta)$ are not simple. We introduce the functions $\chi(x, \zeta)$ and $\bar{\chi}(x, \zeta)$ with the definite analytical and asymptotic properties, in place of $\varphi^{(2)}(x, \zeta)$ and $\psi^{(2)}(x, \zeta)$ according to Kaup [126]. Consider the adjoint equation of (2.1),

$$\frac{\partial f^A}{\partial x} - \frac{i}{2\zeta} \begin{pmatrix} n & 3i\phi & 3\eta \\ -\frac{2}{3}i\zeta\phi^* & 0 & -2i\zeta\phi^* \\ -\frac{\eta}{3} & -i\phi & -n \end{pmatrix} f^A = \begin{pmatrix} -3i\zeta & 0 & 0 \\ 0 & -i\zeta & 0 \\ 0 & 0 & i\zeta \end{pmatrix} f^A. \quad (2.16)$$

We define the solutions of (2.16), $\varphi^{(1)A}(x, \zeta)$ and $\psi^{(1)A}(x, \zeta)$, satisfying the boundary conditions,

$$\varphi^{(1)A}(x, \zeta) e^{3i\zeta x} \rightarrow \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad \varphi^{(2)A}(x, \zeta) e^{i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varphi^{(3)A}(x, \zeta) e^{-i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

as $x \rightarrow -\infty$, (2.17)

$$\psi^{(1)A}(x, \zeta) e^{3i\zeta x} \rightarrow \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \quad \psi^{(2)A}(x, \zeta) e^{i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi^{(3)A}(x, \zeta) e^{-i\zeta x} \rightarrow \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},$$

as $x \rightarrow +\infty$. (2.18)

Then it can be shown that $\varphi^{(1)A}(x, \zeta) e^{3i\zeta x}$ and $\psi^{(3)A}(x, \zeta) e^{-i\zeta x}$ are analytic functions of ζ in the upper half ζ -plane ($\text{Im}(\zeta) > 0$) and $\varphi^{(3)A}(x, \zeta) e^{-i\zeta x}$ and $\psi^{(1)A}(x, \zeta) e^{3i\zeta x}$ are analytic functions of ζ in the lower half ζ -plane ($\text{Im}(\zeta) < 0$).

The Jost functions $\varphi^{(i)}$ and $\psi^{(i)}$ can be represented in terms of $\varphi^{(j)A}$ and $\psi^{(j)A}$:

$$\varphi_n^{(i)} = \frac{1}{2} \sum_{j,k=1}^3 \varepsilon_{ijk} \sum_{m,p=1}^3 \varepsilon_{nmp} \varphi_m^{(j)A} \varphi_p^{(k)A} \alpha_m \alpha_p e^{3i\zeta x}, \quad (2.19a)$$

$$\psi_n^{(i)} = \frac{1}{2} \sum_{j,k=1}^3 \varepsilon_{ijk} \sum_{m,p=1}^3 \varepsilon_{nmp} \psi_m^{(j)A} \psi_p^{(k)A} \alpha_m \alpha_p e^{3i\zeta x}, \quad (2.19b)$$

where ε_{ijk} is the alternating tensor and $\alpha_1 = \frac{1}{3}$, $\alpha_2 = 1$ and $\alpha_3 = -1$. Conversely, it holds that

$$\varphi_n^{(i)A} = \frac{1}{2\alpha_n} \sum_{j,k=1}^3 \varepsilon_{ijk} \sum_{m,p=1}^3 \varepsilon_{nmp} \varphi_m^{(j)} \varphi_p^{(k)} e^{-3i\zeta x}, \quad (2.20a)$$

$$\psi_n^{(i)A} = \frac{1}{2\alpha_n} \sum_{j,k=1}^3 \varepsilon_{ijk} \sum_{m,p=1}^3 \varepsilon_{nmp} \psi_m^{(j)} \psi_p^{(k)} e^{-3i\zeta x}. \quad (2.20b)$$

Substitution of (2.9) and (2.11) into (2.20) shows that

$$\varphi^{(j)A}(x, \zeta) = \sum_{k=1}^3 b_{kj} \psi^{(k)A}(x, \zeta), \quad (2.21a)$$

$$\psi^{(j)A}(x, \zeta) = \sum_{k=1}^3 a_{kj} \varphi^{(k)A}(x, \zeta). \quad (2.21b)$$

We now introduce χ and $\bar{\chi}$ by the relations

$$\chi_n = \sum_{m,p=1}^3 \varepsilon_{nmp} \varphi_m^{(1)A} \psi_p^{(3)A} \alpha_m \alpha_p e^{2i\zeta x}, \quad (2.22a)$$

$$\bar{\chi}_n = \sum_{m,p=1}^3 \varepsilon_{nmp} \varphi_m^{(3)A} \psi_p^{(1)A} \alpha_m \alpha_p e^{2i\zeta x}. \quad (2.22b)$$

Then, χ_n and $\bar{\chi}_n$ are analytic functions of ζ in the upper and the lower half ζ -plane, respectively and $\chi e^{i\zeta x}$ and $\bar{\chi} e^{i\zeta x}$ are proved to be solutions of (2.1). It follows from (2.19), (2.21) and (2.22) that

$$\chi(x, \zeta) = e^{-i\zeta x} (\theta_{21}(\zeta) \psi^{(1)}(x, \zeta) - \theta_{11}(\zeta) \psi^{(2)}(x, \zeta)), \quad (2.23a)$$

$$\bar{\chi}(x, \zeta) = e^{-i\zeta x} (\theta_{33}(\zeta) \psi^{(2)}(x, \zeta) - \theta_{23}(\zeta) \psi^{(3)}(x, \zeta)). \quad (2.23b)$$

By making use of the asymptotic forms of $\varphi^{(i)A}$ and $\psi^{(i)A}$ for large $|\zeta|$, the asymptotic expressions of χ and $\bar{\chi}$ are obtained :

$$\chi(x, \zeta) = - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + O(\zeta^{-1}) \quad \text{for } \text{Im}(\zeta) > 0, \quad (2.24a)$$

$$\bar{\chi}(x, \zeta) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + O(\zeta^{-1}) \quad \text{for } \text{Im}(\zeta) < 0. \quad (2.24b)$$

These relations will be used to obtain an integral representation of

$\psi^{(2)} e^{-i\zeta x}$ in the next section.

Next, we present two kinds of symmetrical property of the scattering matrix. When $f(x, \zeta)$ is a solution of (2.1), $f_A(x, \zeta) \equiv B(\zeta) [f(x, \zeta^*)]^*$ is a solution of (2.16), where

$$B(\zeta) \equiv \begin{pmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{pmatrix} \equiv \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.25)$$

Taking account of the boundary conditions (2.4), (2.5), (2.17) and (2.18), we have

$$\varphi_A^{(m)}(x, \zeta) = B(\zeta) [\varphi^{(m)}(x, \zeta^*)]^* = \alpha_m \beta_m \varphi^{(m)A}(x, \zeta),$$

$$\psi_A^{(n)}(x, \zeta) = B(\zeta) [\psi^{(n)}(x, \zeta^*)]^* = \alpha_n \beta_n \psi^{(n)A}(x, \zeta).$$

It follows from these and (2.9) and (2.21a) that

$$\theta_{nm}(\zeta) = \frac{\gamma_n}{\gamma_m} [a_{mn}(\zeta^*)]^*, \quad (2.26)$$

where $\gamma_m = \alpha_m \beta_m$, $\gamma_1 = 1$, $\gamma_2 = 2\zeta$, $\gamma_3 = -1$.

It is easily seen that when

$$f(x, \zeta) = \begin{pmatrix} f_1(x, \zeta) \\ f_2(x, \zeta) \\ f_3(x, \zeta) \end{pmatrix}$$

is a solution of (2.1), then

$$\tilde{f}(x, \zeta) \equiv \begin{pmatrix} f_3(x, -\zeta) \\ -f_2(x, -\zeta) \\ f_1(x, -\zeta) \end{pmatrix} e^{2i\zeta x}$$

is also a solution of (2.1). Consideration on the boundary conditions leads to the relations

$$\varphi^{(1)}(x, \zeta) = \tilde{\varphi}^{(3)}(x, \zeta), \quad \varphi^{(3)}(x, \zeta) = \tilde{\varphi}^{(1)}(x, \zeta), \quad (2.27a)$$

$$\varphi^{(2)}(x, \zeta) = -\tilde{\varphi}^{(2)}(x, \zeta), \quad (2.27b)$$

$$\psi^{(1)}(x, \zeta) = \tilde{\psi}^{(3)}(x, \zeta), \quad \psi^{(3)}(x, \zeta) = \tilde{\psi}^{(1)}(x, \zeta), \quad (2.27c)$$

$$\psi^{(2)}(x, \zeta) = -\tilde{\psi}^{(2)}(x, \zeta). \quad (2.27d)$$

We get from (2.13) and (2.8) the second symmetrical property :

$$\left. \begin{aligned} a_{11}(\zeta) &= a_{33}(-\zeta), \quad a_{13}(\zeta) = a_{31}(-\zeta), \quad a_{22}(\zeta) = a_{22}(-\zeta), \\ a_{12}(\zeta) &= -a_{32}(-\zeta), \quad a_{21}(\zeta) = -a_{23}(-\zeta). \end{aligned} \right\} \quad (2.28)$$

Let us now consider the time evolution of the scattering matrix. It can be determined by the knowledge of only the asymptotic forms of the potentials. We require that the boundary conditions (2.4) and (2.5) are satisfied at any instants. We write the asymptotic form of (2.2) in $x \rightarrow \pm\infty$ as follows :

$$\frac{\partial f}{\partial t} = \begin{pmatrix} i(\frac{2}{3}\zeta^2 - 2\zeta) & 0 & 0 \\ 0 & -\frac{4}{3}i\zeta^2 & 0 \\ 0 & 0 & i(\frac{2}{3}\zeta^2 + 2\zeta) \end{pmatrix} f + c(\zeta)If, \quad (2.29)$$

where $c(\zeta)$ is a constant depending on ζ and I is the unit matrix.

As pointed out in chap. 1 in connection with (I.2.32), the addition of $c(\zeta)If$ makes no change in the evolution equations (1.1) and (1.5).

Substitution of the asymptotic form of $\varphi^{(1)}$ in $x \rightarrow -\infty$ into (2.29) yields $c(\zeta) = -i(2\zeta^2/3 - 2\zeta)$. If we substitute the asymptotic form of $\varphi^{(1)}$ in $x \rightarrow \infty$, i.e.,

$$\varphi^{(1)}(x, \zeta, t) = \begin{pmatrix} a_{11}(\zeta, t) e^{3i\zeta x} \\ a_{12}(\zeta, t) e^{i\zeta x} \\ a_{13}(\zeta, t) e^{-i\zeta x} \end{pmatrix}$$

into (2.29) with this $c(\zeta)$, then we have

$$\frac{\partial a_{11}}{\partial t} = 0, \quad \frac{\partial a_{12}}{\partial t} = (-2i\zeta^2 + 2i\zeta)a_{12}, \quad \frac{\partial a_{13}}{\partial t} = 4i\zeta a_{13}. \quad (2.30a)$$

Similarly, we obtain

$$\frac{\partial a_{21}}{\partial t} = (2i\zeta^2 - 2i\zeta)a_{21}, \quad \frac{\partial a_{22}}{\partial t} = 0, \quad \frac{\partial a_{23}}{\partial t} = (2i\zeta^2 + 2i\zeta)a_{23}, \quad (2.30b)$$

$$\frac{\partial a_{31}}{\partial t} = -4i\zeta a_{31}, \quad \frac{\partial a_{32}}{\partial t} = (-2i\zeta^2 - 2i\zeta)a_{32}, \quad \frac{\partial a_{33}}{\partial t} = 0. \quad (2.30c)$$

The time evolution of (b_{jk}) is the same as that of (a_{jk}) .

§ 5.3 Solution to the Inverse Problem

We first derive integral representations for $\psi^{(j)}$ ($j=1,2$). Since $\psi^{(3)}$ is not independent of $\psi^{(1)}$ as in (2.27c), the integral representation of $\psi^{(3)}$ is not required. We assume for simplicity the potentials to be on compact support, so that all functions are analytic for all ζ in the complex ζ -plane except for $\zeta=0$. In the more general case of non-compact support, it is needed only that all the contour integrals in the representation obtained so are reduced to integrals along the real axis plus all contributions due to any poles. We define the contour C to be the contour in the complex ζ -plane, extending from $-\infty + i0^+$ to 0^- , then from 0^+ to $+\infty + i0^+$ and passing above all zeros of a_{33} and b_{11} . Similarly, \bar{C} is the contour extending from $-\infty + i0^-$ to 0^- , then from 0^+ to $+\infty + i0^-$ and passing under all zeros of a_{11} and b_{33} (see Fig. 1.).

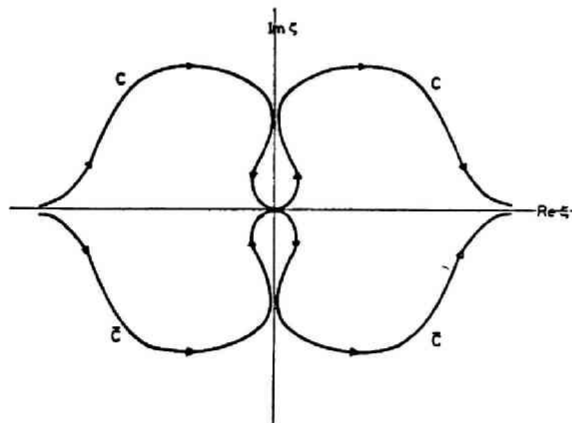


Fig. 1. The contour C and \bar{C} in the complex ζ -plane to obtain integral representations for $\psi^{(1)}$ and $\psi^{(2)}$.

Consider the contour integral

$$\oint \frac{\varphi^{(1)}(x, \zeta') e^{-3i\zeta x}}{a_{11}(\zeta')(\zeta' - \zeta)} d\zeta',$$

for ζ above \bar{C} ; its value is $i\pi \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on account of (2.14c) and (2.15b).

Replacing $\varphi^{(1)}$ by $\sum_k a_{1k} \psi^{(k)}$ and using (2.14a) and (2.27c), we obtain

$$\begin{aligned} \psi^{(1)}(x, \zeta) e^{-3i\zeta x} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2\pi i} \oint \frac{d\zeta'}{\zeta' - \zeta} \frac{a_{12}(\zeta')}{a_{11}(\zeta')} \psi^{(2)}(x, \zeta') e^{-3i\zeta' x} \\ &\quad - \frac{1}{2\pi i} \oint \frac{d\zeta'}{\zeta' - \zeta} \frac{a_{13}(\zeta')}{a_{11}(\zeta')} \tilde{\psi}^{(1)}(x, \zeta') e^{-3i\zeta' x}. \end{aligned} \quad (3.1)$$

Similarly, considering

$$\left[\oint - \oint \right] \frac{d\zeta'}{\zeta' - \zeta} \psi^{(2)}(x, \zeta') e^{-i\zeta' x}$$

for ζ between C and \bar{C} and using (2.23), (2.24) and (2.27c), we get

$$\begin{aligned} \psi^{(2)}(x, \zeta) e^{-i\zeta x} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2\pi i} \oint \frac{d\zeta'}{\zeta' - \zeta} \frac{\theta_{21}(\zeta')}{\theta_{11}(\zeta')} \psi^{(1)}(x, \zeta') e^{-i\zeta' x} \\ &\quad + \frac{1}{2\pi i} \oint \frac{d\zeta'}{\zeta' - \zeta} \frac{\theta_{23}(\zeta')}{\theta_{33}(\zeta')} \tilde{\psi}^{(1)}(x, \zeta') e^{-i\zeta' x}. \end{aligned} \quad (3.2)$$

We now assume that $\psi^{(1)}$ can be represented by

$$\psi^{(1)}(x, \zeta) e^{-3i\zeta x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \int_x^\infty [K(x, s) + \frac{1}{\zeta} L(x, s)] e^{2i\zeta(s-x)} ds, \quad (3.3)$$

where the transformation kernels K and L are column vectors independent of ζ . The structure of the transformation kernels in (3.3) was suggested by Kaup [129], [155]. The necessary and sufficient condition for (3.3) to satisfy (2.1) is given by the following equations :

$$K = \begin{pmatrix} K_1 \\ 0 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_1 \\ L_2 \\ -L_1 \end{pmatrix}, \quad (3.4)$$

$$\frac{\partial K_1}{\partial x} + \frac{\partial K_1}{\partial s} = 4iL_1, \quad (3.5a)$$

$$\frac{\partial L_1}{\partial x} - \frac{\partial L_1}{\partial s} = -\frac{i}{2}nK_1 - \phi^*L_2, \quad (3.5b)$$

$$\frac{\partial L_2}{\partial x} = \frac{1}{2}\phi K_1, \quad (3.5c)$$

$$\lim_{s \rightarrow \infty} K_1(x, s) = 0, \quad \lim_{s \rightarrow \infty} L_1(x, s) = 0, \quad (3.6)$$

$$L_1(x, x) = \frac{i}{4}n, \quad L_2(x, x) = -\frac{1}{2}\phi. \quad (3.7)$$

It can be proved by the method of characteristics that the solution of (3.5) with the boundary conditions (3.6) and (3.7) exists and is unique.

We eliminate $\psi^{(2)}$ from (3.1) and (3.2), substitute (3.3) into the result obtained so and take the Fourier transform with respect to ζ in which the contour passes under $\zeta=0$. Thus we obtain the Gel'fand-Levitan equations for $y > x$,

$$\begin{aligned} K_1(x, y) + 2i \int_y^\infty L_1(x, s) ds + G(x, y) + \int_x^\infty [K_1(x, s)G(s, y) \\ + L_1(x, s)\{Q(s, y) + R(s, y)\}] ds = 0, \end{aligned} \quad (3.8a)$$

$$2i \int_y^\infty L_2(x, s) ds + F(y) + \int_x^\infty L_2(x, s)\{Q(s, y) + R(s, y)\} ds = 0, \quad (3.8b)$$

$$\begin{aligned} -2i \int_y^\infty L_1(x, s) ds + H(x, y) + \int_x^\infty [K_1(x, s)H(s, y) \\ - L_1(x, s)\{Q(s, y) + R(s, y)\}] ds = 0, \end{aligned} \quad (3.8c)$$

where

$$F(x) = \frac{1}{\pi} \oint d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta x}, \quad (3.9a)$$

$$G(x, y) = -\frac{1}{2\pi^2 i} \oint d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta y} \oint \frac{d\zeta'}{\zeta' - \zeta} \frac{b_{21}(\zeta')}{b_{11}(\zeta')} e^{2i\zeta' x}, \quad (3.9b)$$

$$H(x, y) = \frac{1}{\pi} \oint d\zeta \frac{a_{13}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta(x+y)} + \frac{1}{2\pi^2 i} \oint d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta y} \oint \frac{d\zeta'}{\zeta' - \zeta - i\varepsilon} \frac{b_{23}(\zeta')}{b_{33}(\zeta')} e^{-2i\zeta' x}, \quad (3.9c)$$

$$Q(x, y) = -\frac{1}{2\pi^2 i} \oint d\zeta \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta y} \oint \frac{d\zeta'}{\zeta'(\zeta' - \zeta)} \frac{b_{21}(\zeta')}{b_{11}(\zeta')} e^{2i\zeta' x}, \quad (3.9d)$$

$$R(x, y) = \frac{1}{\pi} \oint \frac{d\zeta}{\zeta} \frac{a_{13}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta(x+y)} + \frac{1}{2\pi^2 i} \oint \frac{a_{12}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta y} \oint \frac{d\zeta'}{\zeta'(\zeta' - \zeta - i\varepsilon)} \frac{b_{23}(\zeta')}{b_{33}(\zeta')} e^{-2i\zeta' x}, \quad (3.9e)$$

where the limit $\varepsilon \rightarrow 0^+$ is to be taken.

We now see that a zero of $a_{11}(\zeta)$ corresponds to a soliton solution of equations (1.1) and (1.5). Let $\zeta_0 = \xi - i\eta$ (ξ, η : real, $\eta > 0$) be a zero of $a_{11}(\zeta)$. Then, owing to (2.26) and (2.28), $a_{33}(-\zeta_0) = 0$, $b_{11}(\zeta_0^*) = 0$ and $b_{33}(-\zeta_0^*) = 0$. Here we confine ourselves to the case that the zeros of $a_{11}(\zeta)$ are simple, so that the residue of $a_{12}(\zeta)/a_{11}(\zeta)$ at the pole ζ_0 is $C \equiv a_{12}(\zeta_0)/\dot{a}_{11}(\zeta_0)$. The residues of $b_{21}(\zeta)/b_{11}(\zeta)$ at $\zeta = \zeta_0^*$ and of $b_{23}(\zeta)/b_{33}(\zeta)$ at $\zeta = -\zeta_0^*$ are both $2\zeta_0^* C^*$ in view of (2.26) and (2.28). We take into account the fact that $(b_{jk}(\zeta))$ is the inverse matrix of $(a_{jk}(\zeta))$ and (2.26) and (2.28) to obtain

$$\frac{a_{13}(\zeta)}{a_{11}(\zeta)} = \left[\frac{a_{13}(-\zeta^*)}{a_{11}(-\zeta^*)} \right]^* - \frac{a_{12}(\zeta)}{a_{11}(\zeta)} \frac{b_{23}(\zeta)}{b_{33}(\zeta)}. \quad (3.10)$$

From this, we see that the residue of $a_{13}(\zeta)/a_{11}(\zeta)$ at $\zeta = \zeta_0$ is $-Cb_{23}(\zeta_0)/b_{33}(\zeta_0)$ provided $\zeta_0 \neq -\zeta_0^*$ (i.e., $\zeta_0 \neq 0$).

If $a_{11}(\zeta)$ possesses only one zero ζ_0 and if all of the off-diagonal elements of the scattering matrix are equal to zero on the real axis, we obtain

$$F(x) = 2iC e^{-2i\zeta_0 x},$$

$$G(x, y) = \frac{4i|C|^2 \zeta_0^*}{\zeta_0^* - \zeta_0} e^{2i(\zeta_0^* x - \zeta_0 y)},$$

$$H(x, y) = -\frac{4i|C|^2 \zeta_0^*}{\zeta_0^* + \zeta_0} e^{2i(\zeta_0^* x - \zeta_0 y)},$$

$$Q(x, y) = \frac{4i|C|^2}{\zeta_0^* - \zeta_0} e^{2i(\zeta_0^* x - \zeta_0 y)},$$

$$R(x, y) = \frac{4i|C|^2}{\zeta_0^* + \zeta_0} e^{2i(\zeta_0^* x - \zeta_0 y)},$$

where

$$C = C_0 e^{-2i\zeta_0(\zeta_0 - 1)t}, \quad (C_0: \text{const})$$

here (2.30a) has been used. The Gel'fand-Levitan equations with these kernels can be reduced to easily solvable linear algebraic equations.

As the result we obtain

$$n(x, t) = -\frac{16|C_0|^2 e^{2i(\zeta_0^* - \zeta_0)x}}{(\zeta_0^* + \zeta_0) [1 - (4|C_0|^2 e^{2i(\zeta_0^* - \zeta_0)x} / (\zeta_0^* - \zeta_0)^2 (\zeta_0^* + \zeta_0))]^2}, \quad (3.11a)$$

$$= -4\gamma^2 \operatorname{sech}^2 [2\gamma \{x - (1-2\xi)t - \delta\}] , \quad (\xi > 0) \quad (3.12a)$$

$$\phi(x, t) = \frac{4iC_0\xi_0 e^{-2i\xi_0 x}}{1 - (4|C_0\xi_0|^2 e^{2i(\xi_0^* - \xi_0)x} / (\xi_0^* - \xi_0)^2 (\xi_0^* + \xi_0))} , \quad (3.11b)$$

$$= 2\gamma\sqrt{2\xi} \operatorname{sech} [2\gamma \{x - (1-2\xi)t - \delta\}] e^{2i(\gamma^2\xi^2)t - 2i\xi(x-t) + i\theta} , \quad (\xi > 0) \quad (3.12b)$$

where

$$\delta = \frac{1}{2\gamma} \log \frac{|C_0\xi_0|}{\gamma\sqrt{2\xi}} , \quad e^{i\theta} = i \frac{C_0\xi_0}{|C_0\xi_0|} \quad (3.13)$$

These clarify the correspondence of a zero of $a_{11}(\xi)$ with a soliton solution.

It should be noted that in the case $\xi < 0$, $n(x, t)$ and $\phi(x, t)$ are given by

$$n(x, t) = 4\gamma^2 \operatorname{cosech}^2 [2\gamma \{x - (1-2\xi)t - \delta_s\}] , \quad (3.14a)$$

$$\phi(x, t) = 2\gamma\sqrt{-2\xi} \operatorname{cosech} [2\gamma \{x - (1-2\xi)t - \delta_s\}] e^{2i(\gamma^2\xi^2)t - 2i\xi(x-t) + i\theta_s} , \quad (3.14b)$$

where

$$\delta_s = \frac{1}{2\gamma} \log \frac{|C_0\xi_0|}{\gamma\sqrt{-2\xi}} , \quad e^{i\theta_s} = i \frac{C_0\xi_0}{|C_0\xi_0|} . \quad (3.15)$$

The integrals $\int_{-\infty}^{\infty} |n(x, t)| dx$ and $\int_{-\infty}^{\infty} |\phi(x, t)| dx$ do not converge because of the singularity of $n(x, t)$ and $\phi(x, t)$. Therefore, the inverse scattering method cannot be applied to this case. Nevertheless, (3.14) certainly satisfy the system of equations (1.1) and (1.5).

Next, let us consider the case $\zeta_0 = -\zeta_0^*$ (i.e., $\xi = 0$). In this case, the zeros of $a_{11}(\zeta)$ and $b_{33}(\zeta)$ coalesce at $\zeta = -i\gamma$. Similarly, the zeros of $a_{33}(\zeta)$ and $b_{11}(\zeta)$ coalesce at $\zeta = i\gamma$. If we put $j=1, l=1$ and $j=1, l=3$ in (2.12), we obtained, respectively,

$$a_{11}(\zeta)\theta_{11}(\zeta) + a_{12}(\zeta)\theta_{21}(\zeta) + a_{13}(\zeta)\theta_{31}(\zeta) = 1, \quad (3.16a)$$

$$a_{11}(\zeta)\theta_{13}(\zeta) + a_{12}(\zeta)\theta_{23}(\zeta) + a_{13}(\zeta)\theta_{33}(\zeta) = 0. \quad (3.16b)$$

From these and the symmetrical properties of the scattering matrix, we see that $\zeta = -i\gamma$ is a zero of $a_{12}(\zeta)$ and $b_{23}(\zeta)$ and $a_{13}(-i\gamma) \neq 0$. If $\zeta = -i\gamma$ is a simple zero of $a_{12}(\zeta)$ and $b_{23}(\zeta)$, then the following two cases occur : i) $\dot{a}_{11}(-i\gamma) \neq 0$, $\text{Re} \{ \dot{a}_{11}(-i\gamma) [a_{13}(-i\gamma)]^* \} = 0$ and ii) $\dot{a}_{11}(-i\gamma) = 0$.

For the case i), it holds that

$$F(x) = G(x, y) = Q(x, y) = 0,$$

$$H(x, y) = \frac{1}{\pi} \oint d\zeta \frac{a_{13}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta(x+y)}, \quad R(x, y) = \frac{1}{\pi} \oint \frac{d\zeta}{\zeta} \frac{a_{13}(\zeta)}{a_{11}(\zeta)} e^{-2i\zeta(x+y)},$$

$$\text{Re} \left[\frac{a_{13}(-i\gamma)}{\dot{a}_{11}(-i\gamma)} \right] = 0.$$

The corresponding solution is

$$\eta(x, t) = \frac{-8\gamma p e^{-4\gamma x}}{(1 + (p/2\gamma)e^{-4\gamma x})^2}, \quad (3.17a)$$

$$\phi(x, t) = 0, \quad (3.17b)$$

where p is given by

$$\frac{a_{13}(-i\gamma)}{\dot{a}_{11}(-i\gamma)} = -ip, \quad p = p_0 e^{4\gamma t}, \quad (3.18)$$

and p is real. If p_0 is positive, (3.17a) is written as

$$n(x, t) = -4\gamma^2 \operatorname{sech}^2[2\gamma(x-t-\delta_0)], \quad \delta_0 = \frac{1}{4\gamma} \log \frac{p_0}{2\gamma}. \quad (3.19)$$

For negative p_0 we get a singular soliton like (3.14).

It should be noted that the solution (3.17) is obtained by taking the limit $\xi \rightarrow 0$ as well as $C_0 \rightarrow 0$ in (3.11) so as for $|C_0|^2/\xi$ to be constant.

In the case ii), $\zeta = -i\gamma$ is a double zero of $a_{11}(\zeta)$ and $b_{33}(\zeta)$ and $\zeta = i\gamma$ is a double zero of $a_{33}(\zeta)$ and $b_{11}(\zeta)$. By making use of (3.16) and the symmetrical properties of the scattering matrix, we can show that

$$F(x) = 4i\gamma e^{-2\gamma(x-t)+2i\gamma^2 t},$$

$$G(x, y) = 8i\gamma\gamma^* e^{-2\gamma(x-t)-2\gamma(y-t)},$$

$$Q(x, y) = -\frac{i}{\gamma} G(x, y),$$

$$H(x, y) = [8(\beta - i\gamma\gamma^*) + 16\gamma(\alpha + i\gamma\gamma^*)x + 16\gamma(\alpha - i\gamma\gamma^*)y - 32\gamma(\alpha + 2\gamma\gamma^*)t] e^{-2\gamma(x-t)-2\gamma(y-t)},$$

$$R(x, y) = \left[\frac{8i}{\gamma}(\alpha + \beta) + 16i(\alpha + i\gamma\gamma^*)x + 16i(\alpha - i\gamma\gamma^*)y - 32i(\alpha + 2\gamma\gamma^*)t \right] e^{-2\gamma(x-t)-2\gamma(y-t)},$$

where α and β are real and $\gamma = \dot{a}_{12}(-i\gamma; 0) / \ddot{a}_{11}(-i\gamma; 0)$. Then, the Gel'fand-Levitan equations can be solved to yield the solution,

$$n(x, t) = \frac{32}{D^2} \left\{ [\gamma(\alpha - \beta) - 4\alpha\gamma^2(x-t) + 8\gamma^3\gamma^*t] e^{-4\gamma(x-t)} + [4\alpha^2 + 2(\gamma\gamma^*)^2] e^{-8\gamma(x-t)} \right. \\ \left. + [\alpha^2 + (\gamma\gamma^*)^2] \left[\frac{1}{\gamma}(3\alpha + \beta) + 4\alpha(x-t) - 8\gamma\gamma^*t \right] e^{-12\gamma(x-t)} \right\}, \quad (3.20a)$$

$$\phi(x, t) = \frac{8\gamma\gamma^*}{D} \left[1 + \frac{1}{\gamma}(\alpha - i\gamma\gamma^*) e^{-4\gamma(x-t)} \right] e^{-2\gamma(x-t) + 2i\gamma^2 t}, \quad (3.20b)$$

where

$$D \equiv 1 + \left\{ \frac{2}{\gamma}(\alpha + \beta) + 8\alpha(x-t) - 16\gamma\gamma^*t \right\} e^{-4\gamma(x-t)} - \frac{1}{\gamma^2} [\alpha^2 + (\gamma\gamma^*)^2] e^{-8\gamma(x-t)}, \quad (3.20c)$$

This solution is singular, because for any fixed t , D necessarily vanishes somewhere.

§ 5.4 N-Soliton Solutions

Here, we study the property of the interaction of N solitons.

Let us assume that $a_{11}(\zeta)$ has N zeros in the lower half ζ -plane and all of the off-diagonal elements of the scattering matrix vanish on the real axis of the ζ -plane, that is,

$$a_{11}(\zeta_j) = 0, \quad (j = 1, 2, \dots, N)$$

$$\zeta_j = \xi_j - i\eta_j, \quad (\eta_j > 0, \xi_j > 0)$$

We also assume that all of ζ_j are simple zeros of $a_{11}(\zeta)$ so that the residue of $a_{12}(\zeta)/a_{11}(\zeta)$ at ζ_j is $C_j \equiv a_{12}(\zeta_j)/\dot{a}_{11}(\zeta_j)$. Introduce $p_n(x)$, $q_n(x)$ and $r_n(x)$ by $L_2(x, y) = \sum_n p_n(x) \exp(-2i\zeta_n y)$, $L_1(x, y) = \sum_n q_n(x) \exp(-2i\zeta_n y)$ and $K_1(x, y) = \sum_n r_n(x) \exp(-2i\zeta_n y)$. The Gel'fand-Levitan equations (3.8) are reduced to the system of linear algebraic equations,

$$p_n(x) + \frac{i/C_n \zeta_n}{\xi_n \eta_n} \sum_m \frac{e^{2i(\zeta_n^* - \zeta_m)x}}{\zeta_n^* - \zeta_m} p_m(x) = -2iC_n \zeta_n, \quad (4.1a)$$

$$\begin{aligned} q_n(x) + \zeta_n r_n(x) + \frac{i/C_n \zeta_n}{\eta_n} \sum_m \frac{e^{2i(\zeta_n^* - \zeta_m)x}}{\zeta_n^* - \zeta_m} r_m(x) \\ + \frac{i/C_n \zeta_n}{\xi_n \eta_n} \sum_m \frac{e^{2i(\zeta_n^* - \zeta_m)x}}{\zeta_n^* - \zeta_m} q_m(x) = -\frac{2/C_n \zeta_n}{\eta_n} e^{2i\zeta_n^* x}, \end{aligned} \quad (4.1b)$$

$$\begin{aligned} q_n(x) + \frac{i/C_n \zeta_n}{\xi_n \eta_n} \sum_m \frac{e^{2i(\zeta_n^* - \zeta_m)x}}{\zeta_n^* - \zeta_m} q_m(x) \\ - \frac{1/C_n \zeta_n}{\xi_n} \sum_m \frac{e^{2i(\zeta_n^* - \zeta_m)x}}{\zeta_n^* - \zeta_m} r_m(x) = -\frac{2i/C_n \zeta_n}{\xi_n} e^{2i\zeta_n^* x}. \end{aligned} \quad (4.1c)$$

Here, we do not write down the explicit form of the N-soliton solution,

but we examine the asymptotic behaviours of the solution as $t \rightarrow \pm\infty$.

Putting

$$\tilde{P}_n(x) = P_n(x) \exp(-2i\xi_n x),$$

$$C_n = C_{n0} \exp\{2\gamma_n(1-2\xi_n)t - 2i(\xi_n^2 - \eta_n^2 - \xi_n)t\},$$

we have, from (4.1a),

$$\begin{aligned} \tilde{P}_n(x) + \frac{i|C_{n0}\xi_n|^2}{\xi_n\eta_n} e^{-4\gamma_n\{x-(1-2\xi_n)t\}} \sum_k \frac{1}{\xi_n^* - \xi_k} \tilde{P}_k(x) \\ = -2iC_{n0}\xi_n e^{-2i\xi_n x - 2i(\xi_n^2 - \eta_n^2 - \xi_n)t - 2\gamma_n\{x-(1-2\xi_n)t\}}. \end{aligned} \quad (4.2)$$

Assuming $\xi_1 > \xi_2 > \dots > \xi_N > 0$, we consider the asymptotic forms of the N-soliton solution for $x - (1-2\xi_m)t \equiv y/\eta_m = \text{const}$ as $t \rightarrow \pm\infty$. As $t \rightarrow \infty$,

$$x - (1-2\xi_j)t = x - (1-2\xi_m)t + 2(\xi_j - \xi_m)t \rightarrow \begin{cases} \infty & (j < m), \\ -\infty & (j > m). \end{cases}$$

Therefore, (4.2) reduces to, for $t \rightarrow \infty$,

$$\tilde{P}_n(x) = 0 \quad \text{for } n < m, \quad (4.3a)$$

$$\left(1 + \frac{|C_{m0}\xi_m|^2}{2\xi_m\eta_m^2} e^{-4\gamma_m}\right) \tilde{P}_m = \Phi_m + \frac{i|C_{m0}\xi_m|^2}{\xi_m\eta_m} e^{-4\gamma_m} \sum_{k=m+1}^N \frac{\tilde{P}_k}{\xi_k - \xi_m^*}, \quad (4.3b)$$

$$\sum_{k=m+1}^N \frac{1}{\xi_n^* - \xi_k} \tilde{P}_k = -\frac{1}{\xi_n^* - \xi_m} \tilde{P}_m \quad \text{for } n \geq m+1, \quad (4.3c)$$

where

$$\Phi_m = -2i C_{m0} \zeta_m e^{-2i \zeta_m x - 2i(\zeta_m^2 - \eta_m^2 - \zeta_m)t} e^{-2\theta_m}.$$

Our aim is to obtain $\sum_{k=m}^N \tilde{p}_k$ by solving (4.3b) and (4.3c). By using the technique similar to that by Zakharov and Shabat (see the Appendix in [59]), we can verify that

$$\sum_{k=m}^N \frac{[a_k(\zeta_k)]^*}{\zeta_k - \zeta_l^*} = 1, \quad (\ell = m, \dots, N) \quad (4.4)$$

$$\sum_{k=m}^N \frac{a_\ell(\zeta_\ell) [a_k(\zeta_k)]^*}{(\zeta_j^* - \zeta_k)(\zeta_k - \zeta_\ell^*)} = \delta_{j\ell}, \quad (\ell = m, \dots, N) \quad (4.5)$$

$$\sum_{k=m+1}^N \frac{\alpha_\ell(\zeta_\ell) [\alpha_k(\zeta_k)]^*}{(\zeta_j^* - \zeta_k)(\zeta_k - \zeta_\ell^*)} = \delta_{j\ell}, \quad (\ell = m+1, \dots, N) \quad (4.6)$$

where

$$\alpha_k(\zeta_k) \equiv \frac{\prod_{\ell=m}^N (\zeta_k^* - \zeta_\ell)}{\prod_{\ell=m}^N (\zeta_k^* - \zeta_\ell^*)}, \quad (k = m, \dots, N) \quad (4.7)$$

$$\alpha_k(\zeta_k) \equiv \frac{\prod_{\ell=m+1}^N (\zeta_k^* - \zeta_\ell)}{\prod_{\ell=m+1}^N (\zeta_k^* - \zeta_\ell^*)} = \frac{\zeta_k^* - \zeta_m^*}{\zeta_k^* - \zeta_m} a_k(\zeta_k), \quad (k = m+1, \dots, N) \quad (4.8)$$

where \prod' denotes that the factor equal to zero is omitted from the product.

Multiplying (4.3c) by $[\alpha_\ell(\zeta_\ell)]^* \alpha_n(\zeta_n) / (\zeta_\ell - \zeta_n^*)$ and summing up over n , we get from (4.6)

$$\tilde{p}_\ell = - \sum_{n=m+1}^N \frac{[\alpha_\ell(\zeta_\ell)]^* \alpha_n(\zeta_n)}{(\zeta_\ell - \zeta_n^*)(\zeta_n^* - \zeta_m)} \tilde{p}_m. \quad (4.9)$$

It follows from (4.5) with $j=m$, $m+1 \leq \ell \leq N$ and (4.8) that

$$\sum_{n=m+1}^N \frac{1}{(\zeta_\ell^* - \zeta_n)(\zeta_n - \zeta_m^*)} \frac{\zeta_n - \zeta_m^*}{\zeta_n - \zeta_m} [\alpha_n(\zeta_n)]^* = \frac{[a_m(\zeta_m)]^*}{(\zeta_m - \zeta_m^*)(\zeta_m - \zeta_\ell^*)}.$$

If we take the complex conjugate of this relation and replace ζ_m with ζ_m^* , then we get

$$\sum_{n=m+1}^N \frac{\alpha_n(\zeta_n)}{(\zeta_\ell - \zeta_n^*)(\zeta_n^* - \zeta_m)} = \frac{a_m(\zeta_m^*)}{(\zeta_m - \zeta_m^*)(\zeta_m - \zeta_\ell)}.$$

Substitution of this into (4.9) yields

$$\tilde{p}_\ell = - \frac{[\alpha_\ell(\zeta_\ell)]^*}{(\zeta_m - \zeta_m^*)(\zeta_m - \zeta_\ell)} a_m(\zeta_m^*) \tilde{p}_m. \quad (4.10)$$

From (4.4) for $\ell = m$, we obtain

$$\sum_{k=m+1}^N \frac{[\alpha_k(\zeta_k)]^*}{\zeta_k - \zeta_m} = 1 + \frac{[a_m(\zeta_m)]^*}{\zeta_m^* - \zeta_m},$$

and therefore

$$\sum_{\ell=m+1}^N \tilde{p}_\ell = \frac{a_m(\zeta_m^*)}{\zeta_m - \zeta_m^*} \tilde{p}_m \left(1 + \frac{[a_m(\zeta_m)]^*}{\zeta_m^* - \zeta_m} \right) = \left(-1 + \frac{a_m(\zeta_m^*)}{\zeta_m - \zeta_m^*} \right) \tilde{p}_m. \quad (4.11)$$

It follows from (4.10), (4.5) for $\ell = m$ and (4.8) that

$$\begin{aligned} \sum_{\ell=m+1}^N \frac{\tilde{p}_\ell}{\zeta_\ell - \zeta_m^*} &= - \frac{a_m(\zeta_m^*)}{\zeta_m - \zeta_m^*} \tilde{p}_m \frac{1}{a_m(\zeta_m)} \left(\sum_{\ell=m+1}^N \frac{[\alpha_\ell(\zeta_\ell)]^* a_m(\zeta_m)}{(\zeta_m - \zeta_\ell)(\zeta_\ell - \zeta_m^*)} \right) \\ &= - \frac{\tilde{p}_m}{\zeta_m - \zeta_m^*} \left(\frac{(\zeta_m - \zeta_m^*)^2}{|a_m(\zeta_m)|^2} + 1 \right). \end{aligned} \quad (4.12)$$

If we substitute (4.12) into (4.3b) and use \tilde{p}_m obtained so and (4.11), then we get

$$\begin{aligned}\tilde{p}_m + \sum_{k=m+1}^N \tilde{p}_k &= \frac{a_m(\zeta_m^*)}{\zeta_m - \zeta_m^*} \tilde{p}_m = \frac{\zeta_m - \zeta_m^*}{[a_m(\zeta_m)]^*} \tilde{p}_m \\ &= \frac{(-2i\gamma_m/[a_m(\zeta_m)]^*) \Phi_m}{1 + (|C_m \zeta_m|^2 / 2\zeta_m \gamma_m^2)(4\gamma_m^2 / |a_m(\zeta_m)|^2) e^{-4\theta_m}}.\end{aligned}$$

In the limit $t \rightarrow \infty$ under the condition $y_m = \text{const.}$, we finally obtain

$$\phi \rightarrow 2\gamma_m \sqrt{2\zeta_m} \operatorname{sech} [2\gamma_m \{ \chi - (1 - 2\zeta_m)t - \delta_m^+ \}] e^{\frac{2i(\gamma_m^2 - \zeta_m^2)t - 2i\zeta_m(\chi - t) + i\theta_m^+}{2}}, \quad (4.13a)$$

where

$$\begin{aligned}\delta_m^+ &= \frac{1}{2\gamma_m} \log \frac{|C_m \zeta_m|}{\gamma_m \sqrt{2\zeta_m}} + \frac{1}{2\gamma_m} \log \frac{2\gamma_m}{|a_m(\zeta_m)|}, \\ e^{i\theta_m^+} &= \left(i \frac{C_m \zeta_m}{|C_m \zeta_m|} \right) \left(\frac{|a_m(\zeta_m)|}{[a_m(\zeta_m)]^*} \right).\end{aligned}$$

Similar calculation yields

$$\eta \rightarrow -4\gamma_m^2 \operatorname{sech}^2 [2\gamma_m \{ \chi - (1 - 2\zeta_m)t + \delta_m^+ \}], \quad (4.13b)$$

as $t \rightarrow \infty$ ($y_m = \text{const.}$).

For the case $t \rightarrow -\infty$ ($y_m = \text{const.}$), by introducing $\tilde{a}_k(\zeta_k)$ in place of $a_k(\zeta_k)$ by

$$\tilde{a}_k(\zeta_k) \equiv \frac{\prod_{\ell=1}^m (\zeta_k^* - \zeta_\ell)}{\prod_{\ell=1}^m (\zeta_k^* - \zeta_\ell^*)}, \quad (k=1, \dots, m)$$

we can obtain in the similar way the following asymptotic forms,

$$\phi \rightarrow 2\gamma_m \sqrt{2\xi_m} \operatorname{sech}\{2\gamma_m\{\chi - (1-2\xi_m)t - \delta_m^-\}\} e^{2i(\gamma_m^2 \xi_m^2)t - 2i\xi_m(\chi - t) + i\theta_m^-}, \quad (4.14a)$$

$$n \rightarrow -4\gamma_m^2 \operatorname{sech}^2\{2\gamma_m\{\chi - (1-2\xi_m)t - \delta_m^-\}\}, \quad (4.14b)$$

where

$$\delta_m^- = \frac{1}{2\gamma_m} \log \frac{|C_m \xi_m|}{\gamma_m \sqrt{2\xi_m}} + \frac{1}{2\gamma_m} \log \frac{2\gamma_m}{|\tilde{a}_m(\xi_m)|},$$

$$e^{i\theta_m^-} = \left(i \frac{C_m \xi_m}{|C_m \xi_m|} \right) \left(\frac{|\tilde{a}_m(\xi_m)|}{i[\tilde{a}_m(\xi_m)]^*} \right).$$

It is seen from these that the asymptotic forms (4.13) and (4.14) as $t \rightarrow \pm\infty$ are the same as the forms of single soliton solutions and only the soliton phase δ and the wave phase θ undergo changes due to the interactions of solitons. These changes throughout the whole process of the interactions are given by

$$\Delta\delta_m = \delta_m^+ - \delta_m^- = \frac{1}{2\gamma_m} \sum_{\ell=m+1}^N \log \left| \frac{\xi_m - \xi_\ell}{\xi_m - \xi_\ell^*} \right| - \frac{1}{2\gamma_m} \sum_{\ell=1}^{m-1} \log \left| \frac{\xi_m - \xi_\ell}{\xi_m - \xi_\ell^*} \right|, \quad (4.15a)$$

$$\Delta\theta_m = \theta_m^+ - \theta_m^- = \sum_{\ell=m+1}^N \arg \frac{\xi_m - \xi_\ell}{\xi_m - \xi_\ell^*} - \sum_{\ell=1}^{m-1} \arg \frac{\xi_m - \xi_\ell}{\xi_m - \xi_\ell^*}. \quad (4.15b)$$

The results (4.15) are as if only paired collisions occur, and such a situation is analogous to those in the KdV equation [30]–[33] and the nonlinear Schrödinger equation [59].

We note that according to the remark given in § 5.3, the case can be also included in the above discussions.

§ 5.5 Results based on Perturbation Method

Let us consider first the possibility that a broad packet of the Langmuir field $\phi(x)$ develops to a series of solitons. This depends on whether the zeros of $a_{11}(\zeta)$ exist on the right side of the lower half ζ -plane. Here we study the initial value problem in which $n(x, t=0)=0$ on the assumption that $|\phi(x, t=0)|$ is small enough for the perturbation method to be applied.

Putting $\varphi^{(1)} e^{-3i\zeta x} = F_1(x, \zeta)$, we obtain the integral equations for $F(x, \zeta)$ at $t=0$ from (2.1) and (2.4),

$$F_1(x, \zeta) = 1 - \int_{-\infty}^x \phi^*(x') F_2(x', \zeta) dx', \quad (5.1a)$$

$$F_2(x, \zeta) = \frac{1}{2\zeta} \int_{-\infty}^x e^{-2i\zeta(x-x')} \phi(x') \{F_1(x', \zeta) + F_3(x', \zeta)\} dx' \quad (5.1b)$$

$$F_3(x, \zeta) = \int_{-\infty}^x e^{-4i\zeta(x-x')} \phi^*(x') F_2(x', \zeta) dx', \quad (5.1c)$$

where $\phi(x) \equiv \phi(x, t=0)$. If we retain the terms up to the second order of $|\phi(x)|$ in the successive approximation for (5.1), we have

$$F_1(x, \zeta) \approx 1 - \frac{1}{2\zeta} \int_{-\infty}^x dx' \phi^*(x') \int_{-\infty}^{x'} e^{-2i\zeta(x'-x'')} \phi(x'') dx''.$$

Then, we get

$$\begin{aligned} a_{11}(\zeta) &= \lim_{x \rightarrow \infty} F_1(x, \zeta) \approx 1 - \frac{1}{2\zeta} \int_{-\infty}^{\infty} dx' \phi^*(x') \int_{-\infty}^{x'} e^{-2i\zeta(x'-x'')} \phi(x'') dx'' \\ &= 1 - \frac{1}{2\zeta} \int_{-\infty}^{\infty} dx' \phi(x') \int_{x'}^{\infty} e^{2i\zeta(x'-x'')} \phi^*(x'') dx''. \end{aligned}$$

Taking the average of these two expressions and integrating by parts, we obtain

$$a_{11}(\zeta) \approx 1 - \frac{1}{8i\zeta^2} \sum_{m=0}^{\infty} \frac{1}{(2\zeta)^{2m}} \left\{ 2 \int_{-\infty}^{\infty} |\phi^{(m)}(x)|^2 dx + \frac{1}{2i\zeta} \int_{-\infty}^{\infty} [\phi^{(m)}(x) \phi^{(m+1)*}(x) - \phi^{(m)*}(x) \phi^{(m+1)}(x)] dx \right\}, \quad (5.2)$$

where

$$\phi^{(m)}(x) = \frac{d^m \phi(x)}{dx^m}$$

Furthermore, let us assume $\phi(x)$ to take the form

$$\phi(x) = \varepsilon \Phi(\delta x), \quad (5.3)$$

where ε and δ are small parameters. Then, if $\delta \ll \varepsilon$, in (5.2), the terms containing the derivatives with respect to x can be neglected.

Therefore, we get

$$a_{11}(\zeta) \approx 1 - \frac{1}{4i\zeta^2} \int_{-\infty}^{\infty} |\phi(x)|^2 dx, \quad (5.4)$$

which has a zero on the right side of the lower half ζ -plane, that is, the zero ζ_0 is given by

$$\zeta_0 \approx \frac{1}{2} e^{-\frac{\pi}{4}i} \left(\int_{-\infty}^{\infty} |\phi(x)|^2 dx \right)^{1/2}. \quad (5.5)$$

So, in such cases one soliton will be formed from the initial state consisting of a Langmuir wave only.

Next, let us consider the interaction of a Langmuir soliton with

the empty soliton given by (3.17b) and (3.19) which are initially placed apart from one another by a finite distance ℓ . As seen from (1.1) the depression of the ion density ($n < 0$) works on Langmuir waves as an attractive potential. Therefore, when the empty soliton collides with the Langmuir soliton, there is the possibility that a part of the Langmuir field in the soliton is picked up by the empty soliton and the latter evolves to a Langmuir soliton. In order to study this, we consider the following eigenvalue problem :

$$\left[-iA \frac{\partial}{\partial x} + \frac{1}{\zeta} (V_0 + V_1) + W_1 \right] f = \zeta f, \quad (5.6)$$

where

$$A = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad V_0 = \frac{n_0}{6} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix},$$

$$V_1 = \frac{n_1}{6} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} + \frac{i}{2} \phi_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad W_1 = -\frac{i}{3} \phi_1^* \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix},$$

and

$$n_0(x) = -4\gamma_0^2 \operatorname{sech}^2 2\gamma_0 x, \quad (5.7)$$

$$n_1(x) = -4\gamma_1^2 \operatorname{sech}^2 [2\gamma_1(x-\ell)] \quad (5.8a)$$

$$\phi_1(x) = 2\gamma_1 \sqrt{2\zeta_1} \operatorname{sech} [2\gamma_1(x-\ell)] e^{-2i\zeta_1 x + i\theta_1}. \quad (5.8b)$$

The eigenvalue equation (5.6) is obtained by putting $\phi = \phi_1$, $n = n_0 + n_1$ in (2.1). In the above equations, $n_0(x)$ represents the empty soliton with the corresponding eigenvalue $\zeta_0 = -i\gamma_0$ ($\gamma_0 > 0$), and $n_1(x)$ and $\phi_1(x)$ a

Langmuir soliton with the corresponding eigenvalue $\zeta_1 = \xi_1 - i\gamma_1$ ($\xi_1, \gamma_1 > 0$) which is placed at $x = \ell$. If $V_1 = W_1 = 0$ in (5.6), then the eigenvalue is ζ_0 and the corresponding eigenvector $f^{(0)}$ is given by

$$f^{(0)} = \begin{pmatrix} \frac{e^{\gamma_0 x}}{2} \operatorname{sech} 2\gamma_0 x - \frac{e^{3\gamma_0 x}}{4} \operatorname{sech}^2 2\gamma_0 x \\ 0 \\ \frac{e^{3\gamma_0 x}}{4} \operatorname{sech}^2 2\gamma_0 x \end{pmatrix}. \quad (5.9)$$

This eigenvalue ζ_0 will shift under the effect of V_1 and W_1 . If the shift has the positive real part of the eigenvalue, the empty soliton evolves to a Langmuir soliton through the interaction with the Langmuir soliton (5.8).

We suppose that ℓ is large enough for the overlap of the ion wave (5.7) and the Langmuir soliton (5.8) to be small. We expand f and ζ in (5.6) as

$$\left. \begin{aligned} f &= f^{(0)} + F^{(1)} + F^{(2)} + \dots, \\ \zeta &= \zeta_0 + \Delta\zeta^{(1)} + \Delta\zeta^{(2)} + \dots \end{aligned} \right\} \quad (5.10)$$

Substitution of these into (5.6) gives rise to

$$L_0 f^{(0)} \equiv \left(-iA \frac{\partial}{\partial x} + \frac{V_0}{\zeta_0} - \zeta_0 I \right) f^{(0)} = 0, \quad (5.11a)$$

$$L_0 F^{(1)} = - \left(\frac{1}{\zeta_0} V_1 + W_1 \right) f^{(0)} + \Delta\zeta^{(1)} \left(I + \frac{V_0}{\zeta_0^2} \right) f^{(0)}, \quad (5.11b)$$

$$L_0 F^{(2)} = \left\{ - \left(\frac{1}{\zeta_0} V_1 + W_1 \right) + \Delta\zeta^{(1)} \left(I + \frac{V_0}{\zeta_0^2} \right) \right\} F^{(1)}$$

$$+ \frac{\Delta \zeta^{(1)}}{\zeta_0^2} \left(V_1 - \frac{\Delta \zeta^{(1)}}{\zeta_0} V_0 \right) f^{(0)} + \Delta \zeta^{(2)} \left(I + \frac{V_0}{\zeta_0^2} \right) f^{(0)}, \quad (5.11c)$$

...

where I denotes the unit matrix. The adjoint equation of (5.11a),

$$L_0^A f^A \equiv \left(iA \frac{d}{dx} + \frac{V_0^T}{\zeta_0} - \zeta_0 I \right) f^A = 0, \quad (5.12)$$

is easily solved to give the solution,

$$f^A = \begin{pmatrix} \frac{3}{2} e^{\gamma_0 x} \operatorname{sech}^2 2\gamma_0 x \\ 0 \\ e^{-\gamma_0 x} \operatorname{sech} 2\gamma_0 x - \frac{e^{\gamma_0 x}}{2} \operatorname{sech}^2 2\gamma_0 x \end{pmatrix}, \quad (5.13)$$

where V_0^T is the transposed matrix of V_0 . Taking the inner product of (5.11b) with the adjoint solution (5.13) and using (5.12), we get

$$\Delta \zeta^{(1)} = \frac{\langle f^A, \left(\frac{1}{\zeta_0} V_1 + W_1 \right) f^{(0)} \rangle}{\langle f^A, \left(I + \frac{V_0}{\zeta_0^2} \right) f^{(0)} \rangle}, \quad (5.14)$$

where the inner product of vectors f and g , $\langle f, g \rangle$, is defined by

$$\langle f, g \rangle \equiv \int_{-\infty}^{\infty} \sum_{i=1}^3 f_i g_i dx.$$

It follows from (5.7) ~ (5.9), (5.13) and (5.14) that

$$\Delta \zeta^{(1)} = -i\gamma_1^2 \int_{-\infty}^{\infty} \operatorname{sech}^2 [2\gamma_1(x-l)] \operatorname{sech}^2 [2\gamma_0 x] dx.$$

This is purely imaginary, therefore we must proceed to higher order calculations to obtain the real part of the eigenvalue.

Two other solutions of (5.11a) independent of the solution (5.9) are

$$g = \begin{pmatrix} 0 \\ e^{\eta_0 x} \\ 0 \end{pmatrix} \text{ and } h = \begin{pmatrix} e^{\eta_0 x} [\{2\eta_0 x(1 - \tanh 2\eta_0 x) + 1\} \operatorname{sech} 2\eta_0 x + e^{-2\eta_0 x}] \\ 0 \\ e^{\eta_0 x} [\{2\eta_0 x(1 + \tanh 2\eta_0 x) - 1\} \operatorname{sech} 2\eta_0 x - e^{-2\eta_0 x}] \end{pmatrix}. \quad (5.15)$$

Let us expand $F^{(1)}$ in (5.11b) by means of $f^{(0)}$, g and h as

$$F^{(1)} = p(x) f^{(0)} + q(x) g + r(x) h. \quad (5.16)$$

Substituting this into (5.11b), we get

$$\begin{aligned} \frac{dp}{dx} &= \frac{i\eta_1}{2\zeta_0} e^{-2\eta_0 x} (f_1^{(0)} + f_3^{(0)})(h_1 + h_3) \\ &\quad - i\Delta\zeta^{(1)} e^{-2\eta_0 x} \left\{ (3f_1^{(0)} h_3 + f_3^{(0)} h_1) + \frac{\eta_0}{2\zeta_0^2} (f_1^{(0)} + f_3^{(0)})(h_1 + h_3) \right\}, \end{aligned} \quad (5.17a)$$

$$\frac{dq}{dx} = \frac{\phi_1}{2\zeta_0} e^{-\eta_0 x} (f_1^{(0)} + f_3^{(0)}), \quad (5.17b)$$

$$\begin{aligned} \frac{dr}{dx} &= -\frac{i\eta_1}{2\zeta_0} e^{-2\eta_0 x} (f_1^{(0)} + f_3^{(0)})^2 \\ &\quad + i\Delta\zeta^{(1)} e^{-2\eta_0 x} \left\{ 4f_1^{(0)} f_3^{(0)} + \frac{\eta_0}{2\zeta_0^2} (f_1^{(0)} + f_3^{(0)})^2 \right\}. \end{aligned} \quad (5.17c)$$

It is evident that the right hand sides of (5.17a) and (5.17c) are real and thus $p(x)$ and $r(x)$ are also real. Taking the inner product of (5.11c) with f^A , we can see that the part $\text{pr}^{(0)+\text{rh}}$ of $F^{(1)}$ does not contribute to the real part of $\Delta\zeta^{(2)}$. The equation (5.17b) reduces to

$$g(x) = \frac{i}{4\gamma_0} \int_{-\infty}^x \phi_1(x) \operatorname{sech} 2\gamma_0 x dx.$$

Finally, we obtain

$$Re(4\dot{\zeta}^{(2)}) = \frac{Re(\langle f^A, W, g(x)g \rangle)}{\langle f^A, (I + \frac{V_0}{\zeta_0^2}) f^{(0)} \rangle} = \frac{1}{8} \left| \int_{-\infty}^{\infty} \phi_1(x) \operatorname{sech} 2\gamma_0 x dx \right|^2. \quad (5.18)$$

The equation (5.18) determines the velocity of the Langmuir soliton to which the empty soliton evolves.

§ 5.6 Constants of Motion

The time invariance of $a_{11}(\zeta)$ permits us to have an infinite number of constants of motion. If we define $F = \varphi^{(1)} e^{-3i\zeta x}$, then we obtain from (2.1)

$$\frac{\partial F_1}{\partial x} = -\frac{i}{2\zeta} n(F_1 + F_3) - \phi^* F_2, \quad (6.1a)$$

$$\frac{\partial F_2}{\partial x} = \frac{1}{2\zeta} \phi(F_1 + F_3) - 2i\zeta F_2, \quad (6.1b)$$

$$\frac{\partial F_3}{\partial x} = \frac{i}{2\zeta} n(F_1 + F_3) + \phi^* F_2 - 4i\zeta F_3. \quad (6.1c)$$

If we eliminate F_2 and F_3 from these, we get

$$\begin{aligned} 16\zeta^3 \frac{\partial F_1}{\partial x} = & 4i\zeta^2 \left\{ 3 \frac{\partial^2 F_1}{\partial x^2} + \frac{1}{n} (2i|\phi|^2 - 3 \frac{\partial \eta}{\partial x}) \frac{\partial F_1}{\partial x} - 2n F_1 \right\} \\ & + 2\zeta \left\{ \frac{\partial^3 F_1}{\partial x^3} + \frac{1}{n\phi^*} (2n \frac{\partial \phi^*}{\partial x} - 4\phi^* \frac{\partial \eta}{\partial x} + 3i\phi^* |\phi|^2) \frac{\partial^2 F_1}{\partial x^2} + \frac{1}{n\phi^*} (\phi^* \frac{\partial^2 \eta}{\partial x^2} \right. \\ & \left. - 2n \frac{\partial^2 \phi^*}{\partial x^2} + 2 \frac{\partial \eta}{\partial x} \frac{\partial \phi^*}{\partial x} - 2\phi^* n^2 - 2i|\phi|^2 \frac{\partial \phi^*}{\partial x} - i\phi^* \frac{\partial |\phi|^2}{\partial x}) \frac{\partial F_1}{\partial x} \right\} \\ & + \frac{1}{n\phi^*} (i\phi^* \frac{\partial \eta}{\partial x} - i n \frac{\partial \phi^*}{\partial x} + \phi^* |\phi|^2) \frac{\partial^3 F_1}{\partial x^3} + \frac{1}{n\phi^*} (i n \frac{\partial^2 \phi^*}{\partial x^2} - i\phi^* \frac{\partial^2 \eta}{\partial x^2} - |\phi|^2 \frac{\partial \phi^*}{\partial x} \\ & - \phi^* \frac{\partial |\phi|^2}{\partial x}) \frac{\partial^2 F_1}{\partial x^2} + \frac{1}{n\phi^{*2}} \left\{ i\phi^* \frac{\partial \phi^*}{\partial x} \frac{\partial^2 \eta}{\partial x^2} - i\phi^* \frac{\partial^2 \phi^*}{\partial x^2} \frac{\partial \eta}{\partial x} - \phi^* |\phi|^2 \frac{\partial^2 \phi^*}{\partial x^2} \right. \\ & \left. + |\phi|^2 \left(\frac{\partial \phi^*}{\partial x} \right)^2 + \phi^* \frac{\partial \phi^*}{\partial x} \frac{\partial |\phi|^2}{\partial x} - 2i\phi^{*2} n \frac{\partial \eta}{\partial x} + 2in^2 \phi^* \frac{\partial \phi^*}{\partial x} - 2n\phi^{*2} |\phi|^2 \right\} \frac{\partial F_1}{\partial x} \\ & + \frac{1}{n\phi^*} \left\{ 2i\phi^* n \frac{\partial^2 \eta}{\partial x^2} - 2in^2 \frac{\partial^2 \phi^*}{\partial x^2} - 4i\phi^* \left(\frac{\partial \eta}{\partial x} \right)^2 + 4in \frac{\partial \eta}{\partial x} \frac{\partial \phi^*}{\partial x} - 6\phi^* |\phi|^2 \frac{\partial \eta}{\partial x} \right\} \end{aligned}$$

$$+ 4\eta|\phi|^2 \frac{\partial \phi^*}{\partial x} + 2\eta\phi^* \frac{\partial |\phi|^2}{\partial x} + 2i\phi^* |\phi|^4 \} F_1. \quad (6.2)$$

From (2.4) and (2.13),

$$F_1 \rightarrow 1 \text{ as } x \rightarrow -\infty \text{ and } F_1 \rightarrow a_{11}(\zeta) \text{ as } x \rightarrow +\infty.$$

Therefore, introducing $c(x, \zeta; t)$ by the relation $F_1 = e^c$, we obtain the following asymptotic forms :

$$\left. \begin{aligned} c &\rightarrow 0 \text{ as } x \rightarrow -\infty, \\ c &\rightarrow \ln a_{11}(\zeta) \text{ as } x \rightarrow +\infty. \end{aligned} \right\} \quad (6.3)$$

The equation (6.2) makes it possible to calculate the coefficients of the asymptotic expansion of the function $\partial c / \partial x$ in powers of ζ^{-1} ,

$$\frac{\partial c}{\partial x} \sim \sum_{n=1}^{\infty} \frac{u_n}{(2i\zeta)^n}. \quad (6.4)$$

The recurrence formula is given by the equations

$$u_1 = \eta, \quad u_2 = \frac{1}{i} |\phi|^2, \quad u_3 = -\frac{1}{2} \left(\eta^2 + \frac{2}{i} \phi^* \frac{\partial \phi}{\partial x} \right), \quad (6.5a)$$

$$\begin{aligned} 2u_{n+3} = & -3 \frac{\partial u_{n+2}}{\partial x} - \frac{\partial^2 u_{n+1}}{\partial x^2} - 3 \sum_{j+k=n+2} u_j u_k - 3 \sum_{j+k=n+1} u_j \frac{\partial u_k}{\partial x} \\ & - \sum_{j+k+l=n+1} u_j u_k u_l + \frac{1}{\eta} \left(3 \frac{\partial \eta}{\partial x} - 2i |\phi|^2 \right) u_{n+2} \\ & - \frac{1}{\eta \phi^*} \left(2\eta \frac{\partial \phi^*}{\partial x} - 4\phi^* \frac{\partial \eta}{\partial x} + 3i\phi^* |\phi|^2 \right) \left(\frac{\partial u_{n+1}}{\partial x} + \sum_{j+k=n+1} u_j u_k \right) \\ & + \frac{1}{\eta \phi^*} \left(2\eta \frac{\partial^2 \phi^*}{\partial x^2} + 2\phi^* \eta^2 - \phi^* \frac{\partial^2 \eta}{\partial x^2} - 2 \frac{\partial \eta}{\partial x} \frac{\partial \phi^*}{\partial x} + 2i |\phi|^2 \frac{\partial \phi^*}{\partial x} + i \phi^* \frac{\partial |\phi|^2}{\partial x} \right) u_{n+1} \\ & + \frac{1}{\eta \phi^*} \left(\phi^* \frac{\partial \eta}{\partial x} - \eta \frac{\partial \phi^*}{\partial x} - i \phi^* |\phi|^2 \right) \left(\frac{\partial^2 u_n}{\partial x^2} + 3 \sum_{j+k=n} u_j \frac{\partial u_k}{\partial x} + \sum_{j+k+l=n} u_j u_k u_l \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{n\phi^*} \left(n \frac{\partial^3 \phi^*}{\partial x^2} - \phi^* \frac{\partial^3 \eta}{\partial x^2} + i|\phi|^2 \frac{\partial \phi^*}{\partial x} + i\phi^* \frac{\partial |\phi|^2}{\partial x} \right) \left(\frac{\partial u_n}{\partial x} + \sum_{j+k=n} u_j u_k \right) \\
& + \frac{1}{n\phi^{*2}} \left\{ \phi^* \frac{\partial \phi^*}{\partial x} \frac{\partial^2 \eta}{\partial x^2} - \phi^* \frac{\partial^2 \phi^*}{\partial x^2} \frac{\partial \eta}{\partial x} + i\phi^* |\phi|^2 \frac{\partial^2 \phi^*}{\partial x^2} - i|\phi|^2 \left(\frac{\partial \phi^*}{\partial x} \right)^2 - i\phi^* \frac{\partial \phi^*}{\partial x} \frac{\partial |\phi|^2}{\partial x} \right. \\
& \left. - 2\phi^{*2} n \frac{\partial \eta}{\partial x} + 2n^2 \phi^* \frac{\partial \phi^*}{\partial x} + 2in\phi^{*2} |\phi|^2 \right\} u_n. \quad (n=1, 2, 3, \dots) \quad (6.5b)
\end{aligned}$$

In view of (2.15b), the function $\ln a_{11}(\zeta)$ also admits of an asymptotic expansion in powers of ζ^{-1} :

$$\ln a_{11}(\zeta) \sim \sum_{n=1}^{\infty} \frac{C_n}{\zeta^n}. \quad (6.6)$$

From (6.3), (6.4) and (6.6), it follows that

$$(2i)^n C_n = \int_{-\infty}^{\infty} u_n(x, t) dx. \quad (n=1, 2, \dots) \quad (6.7)$$

Since $a_{11}(\zeta)$ is time invariant, (6.7) are the constants of motion.

The first five of the constants of motion are given by

$$\begin{aligned}
2iC_1 &= \int_{-\infty}^{\infty} n(x, t) dx, \quad (2i)^2 C_2 = \frac{1}{i} \int_{-\infty}^{\infty} |\phi(x, t)|^2 dx, \\
(2i)^3 C_3 &= -\frac{1}{2} \int_{-\infty}^{\infty} \left\{ n^2 + \frac{1}{i} \left(\phi^* \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi^*}{\partial x} \right) \right\} dx, \\
(2i)^4 C_4 &= \int_{-\infty}^{\infty} \left\{ 2in|\phi|^2 + i \left| \frac{\partial \phi}{\partial x} \right|^2 \right\} dx, \\
(2i)^5 C_5 &= \int_{-\infty}^{\infty} \left\{ \frac{1}{2} n^3 + \frac{1}{8} \left(\frac{\partial \eta}{\partial x} \right)^2 + \frac{3}{2} |\phi|^4 - \frac{3}{2} in \left(\phi^* \frac{\partial \phi}{\partial x} - \phi \frac{\partial \phi^*}{\partial x} \right) \right. \\
& \quad \left. + \frac{i}{2} \left(\phi^* \frac{\partial^3 \phi}{\partial x^3} - \phi \frac{\partial^3 \phi^*}{\partial x^3} \right) \right\} dx.
\end{aligned}$$

Here partial integration has been used to obtain the symmetrical forms in the above.

Now let us consider the N-soliton solution. In this case, in view of the analyticity properties and the asymptotic form, we see that $a_{11}(\zeta)$ has the following form :

$$a_{11}(\zeta) = \prod_{p=1}^N \frac{\zeta - \zeta_p}{\zeta - \zeta_p^*},$$

where $\zeta_p = \xi_p - i\gamma_p$ ($\xi_p, \gamma_p > 0$, $\zeta_p \neq \zeta_g$ ($p \neq g$), $p=1, 2, \dots, N$) are zeros of $a_{11}(\zeta)$ characterizing the N-soliton solution. From this we can easily obtain the following relation for N-soliton solutions,

$$C_n = \frac{1}{n} \sum_{p=1}^N (\zeta_p^{*n} - \zeta_p^n).$$

The fact that the system of equations (1.1) and (1.5) has an infinite number of constants of motion is consistent with the character that the identities of solitons are preserved through the mutual interactions as presented in § 5.4. This also implies that the break-up of solitons and the fusion of solitons cannot occur in this simplified system though can occur in the original system of equations (1.1) and (1.2). This seems to relate to that Hirota's method [174] is applicable to this simplified system but not to the original system [201].

Finally, we present remarks on applicability of the system of equations (1.1) and (1.5). When the propagation velocity of the Langmuir soliton (1.3) tends to unity (i.e., sound velocity) keeping the amplitude of E non-zero, the ion sound wave n becomes infinitely large. This divergence is caused by that the nonlinear effect of the ion wave is not taken

into account in deriving (1.1) and (1.2) [202]. In this sense the solutions which we have obtained cannot be compared with phenomena in real plasmas. However, the system of equations (1.1) and (1.5) presents an interesting example which can be solved exactly by the inverse scattering method.

Chapter 6

Concluding Remarks

In the present chapter, the results obtained in the previous chapters are summarized and some related remarks are presented.

In chapter 2, the reductive perturbation method for long waves has been generalized to the systems in which there exist n "quasi-simple" waves each of which belongs to one of the n different families of characteristics. In this new perturbation method, the phase variables are introduced to describe a part of the effects of the mutual-interaction of quasi-simple waves, which permit us to obtain the simple nonlinear equation such as the KdV equation or the Burgers equation for each quasi-simple wave by the non-secularity condition.

First, the method has been developed for the head-on collision of two solitary waves in the Boussinesq equation (II.1.1). Asymptotically, only the phases of them change owing to the head-on collision. The wave forms change only in the overlapping region. In an asymptotic sense, the solution can be described by the superposition of the two solitary waves with the coordinates involving the phase variables.

Next, the method has been extended to the system in which there exist n interacting quasi-simple waves belonging to different families of characteristics. The system can be described in an asymptotic sense by the superposition of the quasi-simple waves governed by their respective KdV (or Burgers) equation. The effect of the mutual-interaction of them is included in the phase variables.

The method has been applied to ion acoustic waves in collisionless

plasmas and shallow water waves and in particular, the phase shifts due to the head-on collision of two solitary waves have been calculated. As a result we see that the Boussinesq equation (II.1.1) cannot describe the interaction of two waves propagating in opposite directions in shallow water. It should be noted that the solitary wave solution of the Boussinesq equation (II.1.1) is unstable to infinitesimal perturbations [203]. However, this does not necessarily mean physical instability of shallow water solitary waves but rather due to the use of an incorrect approximation in deriving the equation from the original system of equations. The non-linear stability analysis for this equation has not been done and is an interesting future problem. On the other hand, the solitary wave (soliton) solution of the KdV equation is stable [204],[205]. Therefore, the present method is to be applied (to only) the systems in which solitary waves are stable.

The results of the computer simulation of ion acoustic waves can be explained well by the present perturbation method [188]. On the other hand, the experimental results of the head-on collision of two shallow water solitons are explained only qualitatively [189]. The method has been also applied successfully to a compressible, viscous and heat-conducting fluid [187].

There are attempts to describe a nonlinear dispersive system by an ensemble of solitons [206],[30]. They are made in the framework of the KdV equation. However, before such attempts, the study presented in chapter 2 is required.

In chapter 3, the reductive perturbation method for strongly dispersive systems has been extended to the system in which two modulated plane waves

interact each other. The method used there is similar to that in chapter 2. In addition to the phase variables for the modulated wave packet, the phase variables for their carrier waves are introduced. The perturbation method has been first developed for the nonlinear Klein-Gordon equation and next, has been shown to be applicable to a general system. It has been shown that the effects of the mutual-interaction of two modulated waves are the shifts in their positions and those in the phases of their carrier waves. The solution is described by the superposition of the two modulated waves which are governed by their respective nonlinear Schrödinger equations.

In particular, the mutual-interaction of two envelope solitons has been investigated on the basis of the nonlinear Klein-Gordon equation. They pass through each other without change of their forms and velocities. This has been verified by the experiments on deep water waves [191]. Here it should be noted that envelope solitons are, in general, unstable to the two-dimensional infinitesimal perturbations in which the transverse linear dimension is sufficiently larger than the longitudinal one [207], [208]. Therefore, such experiments are to be made in a relatively narrow water channel.

In chapter 4, what class of nonlinear evolution equations can be solved exactly by the direct and inverse scattering problem of the third order eigenvalue equation has been investigated. As a result the coupled system of equations (IV.2.17) has been found to be solved exactly by means of the inverse scattering method. This system of equations is derived from the system of equations (V.1.1) and (V.1.2), which describes the interaction between Langmuir waves and ion sound waves in a plasma, on

the assumption that the waves propagate with a speed close to the ion sound speed in the positive x -direction. The both system of equations have solitary wave solutions.

In chapter 5, the system of equations (IV.2.17) has been solved exactly by solving the direct and inverse scattering problem of the associated ~~their~~^{third} order eigenvalue equation. A zero of one of the diagonal elements of the scattering matrix corresponds to a soliton solution. The asymptotic forms of the N -soliton solution are obtained. The wave forms and the velocities of the solitons do not change owing to the interactions between them. This interaction property is similar to that in the KdV equation and that in the nonlinear Schrödinger equation.

If in the initial state consisting of only a Langmuir wave, the amplitude is small and the product of the amplitude and the characteristic length of spatial variation is sufficiently larger than unity, then one soliton is formed. The process that a part of the Langmuir field in a soliton is picked up by a negative amplitude ion sound wave is investigated by means of a perturbation method.

The existence of an infinite number of constants of motion is shown by giving the procedure for obtaining them successively. On the other hand, for the system of equations (V.1.1) and (V.1.2), only three conservation laws have been found. In this system of equations, the interesting phenomena such as the break-up of solitons and the fusion of solitons can occur, while in the system (IV.2.17), they cannot occur. Therefore, the system of equations (V.1.1) and (V.1.2) has no infinite number of conservation laws probably.

We have also found some singular solutions. However, these solutions

will not evolve from bounded localized initial data because of the existence of an infinite number of constants of motion.

The contribution of the continuous spectrum to the solutions, the Bäcklund transformation and the Hamiltonian structure have not been considered here. They are future problems.

The nonlinear system to which the inverse scattering method is applicable — integrable equation appears to be rather special. However, ^{there}_{ar} many nonintegrable systems which are, in some sense, close to the integrable system. These nonintegrable systems can be analyzed approximately by regarding a solution of the integrable system as the zeroth order solution. These analyses can be also carried out in the framework of the inverse scattering method [209]–[211]. In such a sense, it is important to find the new examples of integrable system.

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List of Published Papers

The contents of the present thesis originate in the following published papers :

- [1] M.Oikawa and N.Yajima : "Interactions of solitary waves — A perturbation approach to nonlinear systems," J. Phys. Soc. Japan 34 (1973) 1093. (Chapter 2)
- [2] M.Oikawa and N.Yajima : "A perturbation approach to nonlinear systems. II. Interaction of nonlinear modulated waves," J. Phys. Soc. Japan 37 (1974) 486. (Chapter 3)
- [3] M.Oikawa and N.Yajima : "Generalization of the reductive perturbation method to multi-wave systems," Progr. theor. Phys. Suppl. No. 55 (1974) 36. (Chapters 2 and 3)
- [4] N.Yajima and M.Oikawa : "A class of exactly solvable nonlinear evolution equations," Progr. theor. Phys. 54 (1975) 1576. (Chapter 4)
- [5] N.Yajima and M.Oikawa : "Formation and interaction of sonic-Langmuir solitons — Inverse scattering method," Progr. theor. Phys. 56 (1976) 1719. (Chapter 5)

